A NEW ALGEBRA FOR SCHOOLS

PARTS I AND II

BY

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D. B. TARAPOREVALA SONS & CO. PRIVATE LTD.

210, Dr. DADABHAI NAOROJI ROAD, BOMBAY 1.

1934

PART I. First published August 1934.
PART II. First published August 1934.

Printed and published by R. J. Taraporevala for D. B. Taraporevala Sons & Co. Private Ltd., 210, Dr. Dadabhai Naoroji Road, Bombay 1, at their Electrographic Industries Division, Apte Industrial Estate, Worli, Bombay 18.

PREFACE

THIS book represents an attempt to gather up and make full use of the numerous detailed suggestions on methods of teaching Algebra, on the choice of subject-matter, on the selection of illustrations, on the construction of exercises and test-papers, etc., which have come under the author's notice since he first began (fifteen years ago) to write on the subject. It is divided into three parts. Part I deals with notation, formulae, simple equations and problems ; Part II includes factors, fractions, simultaneous and quadratic equations ; Part III completes the course for "additional mathematics" in School Certificate. Higher Certificate and Scholarship work is dealt with in *Advanced Algebra* (Durell and Robson).

The book, both as regards text and exercises, is written to meet the requirements of ordinary pupils. If a book includes enough (and sufficiently difficult) examples to occupy and train pupils of special ability, it must contain much that is neither required by nor is suitable for many of the others. For this reason, an appendix has been compiled, consisting of revision exercises, harder supplementary exercises and harder test-papers ; and references to it have been inserted at appropriate places. This appendix will also be of use to those teachers who like to have a large range of examples from which to make their own selection for class work, and it supplies in a systematic form the material for a revision course. Both Parts I-II (bound together) and Part III may be obtained with or without the relevant portions of the appendix. Another feature to which the author attaches importance is the provision of groups of "Extra Practice" exercises at the end of Part I and of Part II for pupils who need additional "drill"; similar exercises are included in the appendix to Part III.

The initial difficulties in Algebra are mainly due to the novelty of the notation. These are best overcome by training the pupil to think in numbers when using letters and by demonstrating the practical utility of the notation by applications to formulae. These two principles have determined the selection of the subject-matter of the early chapters. Throughout the book illustrations

have been drawn from practical geometry, physics and mechanics to increase the interest in the theory and to secure variety in its problems, and this object is furthered by a free use of diagrams.

There are a few exercises in Part III (Ex. VIII. *a*, *c*, IX. *a*, *b*) which merely elaborate types of examples included in Part II. The requirements of certain examinations make it necessary to insert them, but the continuity of the course will not be broken if they are passed over.

The author acknowledges gratefully the valuable help he has received from Miss E. M. Read, Mr. G. Ayres, and Mr. A. Buckley, who have read the proofs and have made many useful suggestions. He is also indebted to Mr. R. M. Wright for some of the test-papers and for assistance with the proofs.

C. V. D.

CONTENTS

PART I

*APPENDIX (Parts I.-II.)**

*** These details apply to the edition with Appendix. The book, as noted in the Preface, is available either with or without Appendix. Both editions are issued with Answers or without Answers.**

PART I

CHAPTER I

FIRST NOTIONS

General Statements

THE following example, illustrating the use of letters to generalise statements, is intended for oral discussion.

Example 1. Part of a post is painted black, namely the shaded portion in the figures below; the rest is white. What is the length of the white portion?

The post is 9 ft. long, and the black portion is 7 ft. long;

∴ the white portion is 9 ft. − 7 ft. long or 2 ft. long.

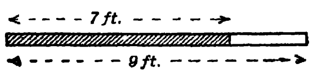


FIG. 1.

Repeat with the following figures:

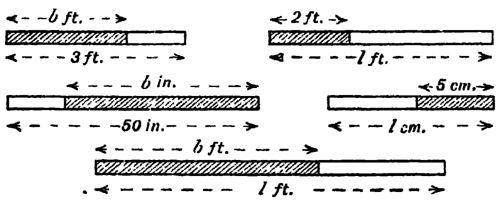


FIG. 2.

The general statement is as follows:

If the length of the post is l feet and if the length of the part painted black is b feet, then the length of the remainder, painted white, is l feet − b feet.

These statements can be shortened by using *brackets*.

Instead of 9 ft. − 7 ft., we may write (9 − 7) ft.; and instead of l feet − b feet, $(l - b)$ feet. The contents of a bracket are then regarded as equivalent to a single number.

Letters are used to represent **numbers.** *Do* **not** *use them to represent quantities, i.e. numbers-of-things.*

Do not say the length of a rod is l, but say that its length is l inches or l feet or l cm., etc.

The Use of Symbols

The symbols $+$, $-$, $\times$, $\div$ have the same meanings in Algebra as in Arithmetic.

The following symbols are in common use :

$=$ means " is equal to " ; thus $5-2=3$ and $4\times5=20$.

$\therefore$ means " therefore " ; thus 1 yd. $=3$ ft., $\therefore$ 4 yd. $=4\times3$ ft.

$>$ means " is greater than " ; thus $5>2$ and $3\frac{1}{2}>2\frac{3}{4}$.

$<$ means " is less than " ; thus $2<5$ and $2\frac{3}{4}<3\frac{1}{2}$.

There are two other symbols it is often convenient to use.

$\simeq$ means " is approximately equal to " ; thus $3\frac{1}{7}\simeq3\cdot14$.

$\neq$ means " is not equal to " ; thus, if $x=5$ and $y=2$, $x\neq y$.

Do not confuse $=$ with $\therefore$; use the symbol $=$ as a verb.

EXERCISE I. a

State in words the following:

1. $3\times4=12$.	2. $7>4$.	3. $5<8$.	4. $1\frac{1}{4}=1\cdot25$.
5. $3<4<5$.	6. $10>8>7$.	7. $\frac{10}{3}=3\frac{1}{3}>3\cdot3$.	8. $1\cdot6<1\frac{2}{3}$.
9. $\mathsf{N}>3$.	10. $\mathsf{A}<8$.	11. $x=y=3$.	12. $a\neq2$.
13. $5\frac{2}{7}\simeq5\cdot3$.	14. $6>z$.	15. $5<b<7$.	16. $4>l>3$.
17. $x\neq0$.	18. $\pi\simeq3\cdot14$.	19. $a\neq b$.	20. $\sqrt{2}\simeq1\cdot41$.

Write in symbols the following :

21. 3^2 is greater than 3.
22. $(\frac{1}{2})^2$ is less than $\frac{1}{2}$.
23. N is equal to 8.
24. A is greater than 5.
25. z is not equal to nought.
26. l is less than $3\frac{1}{2}$.
27. y lies between 10 and 20.
28. A and B are equal.
29. Of the two numbers x and y, the greater is x.
30. An approximation for π is $\frac{22}{7}$.
31. The numbers a and b each equal 10.
32. The number N is less than 100 and is greater than C.
33. Twice N equals fourteen, therefore N equals seven.
34. If N plus three equals eight, then N equals five.
35. If t minus four equals six, then t equals ten.

EXERCISE I. b

1. How much can the water-level rise in each of the glasses shown in Fig. 3 before the water overflows? Answer the same

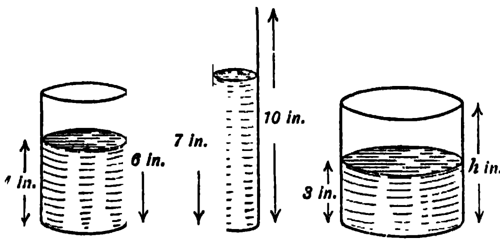


FIG. 3.

question for a glass (i) 5 in. high containing water d in. deep, (ii) h in. high with water d in. deep.

2. There are two parcels of unequal weights in the scale pans of a weighing machine, the heavier on the left.

	(i)	(ii)	(iii)	(iv)	(v)
Weight in left pan - -	6 lb.	10 lb.	8 lb.	W lb.	W lb.
Weight in right pan - -	4 lb.	3 lb.	w lb.	$2\frac{1}{2}$ lb.	w lb.

What weight must be placed in the right-hand scale pan to make them balance? Make a general statement.

3. Part of a rod, see Fig. 4, is painted red, another part is white and the rest is black.

	(i)	(ii)	(iii)	(iv)	(v)
Total length of rod in cm.	10	12	18	l	l
Length of red part in cm. -	5	7	$3\frac{1}{2}$	4	r
Length of white part in cm.	3	4	$4\frac{1}{4}$	7	w

Find the length of the black part in each case. Make a general statement.

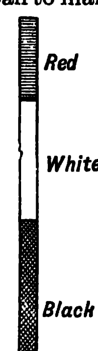


FIG. 4.

4. Find the height of a pile of equal note-books:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Number of note-books - -	10	12	20	n	15	n
Thickness of each book in cm. -	2	3	$1\frac{1}{2}$	2	t	t

5. Write down, without multiplying out,

(i) the number of feet in 2 yd., 7 yd., $4\frac{1}{2}$ yd., x yd.

(ii) the number of pence in 4s., 13s., $3\frac{1}{4}$s., Ps.

6. Fig. 5 gives the dimensions in yards of three rectangular fields; find the total length of fencing required for each field.

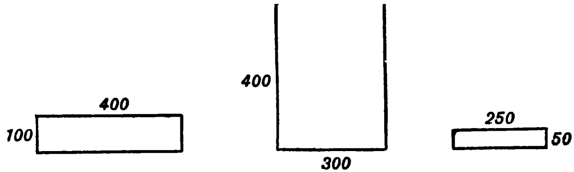


FIG. 5

Give a general statement for the perimeter, when the length is l yd. and the breadth is b yd.

7. (i) With the data of Fig. 6, state the distances of P and Q from B. What is the distance of R from B ?

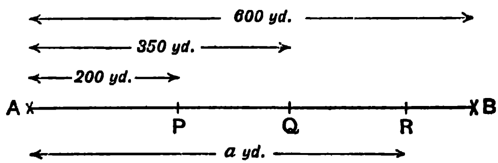


FIG. 6.

(ii) If AB $= s$ yd., AR $= 480$ yd., how far is R from B ?

(iii) If AB $= s$ yd., AR $= a$ yd., how far is R from B ?

8. Find how much a man saves each year :

	(i)	(ii)	(iii)	(iv)	(v)
Income in £ - -	500	700	I	800	I
Expenditure in £ -	400	300	500	E	E

Make a general statement.

9. Tickets for a concert cost two shillings each. What are the receipts if (i) 100, (ii) 240, (iii) n tickets are sold ?

What are the receipts if t tickets are sold at P shillings each ?

10. Write down the size of each unmarked angle in Fig. 7.

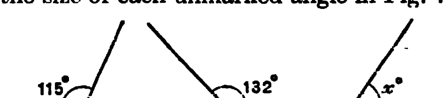


FIG. 7.

Make a general statement.

11. A charabanc holds 40 people. How many are required for a party containing 120 people, 200 people, 480 people ? Make a general statement.

12. A watch loses 10 seconds an hour ; how much does it lose in 3 hours, 8 hours, $2\frac{3}{4}$ hours ? Make a general statement.

13. Fig. 8 represents a rectangular shed EDGF in the corner of a rectangular garden ; the dimensions are given in feet.

Find the lengths of AE and CG (i) if $b=10$, $l=15$, (ii) if $b=12$, $l=20$, (iii) in terms of b and l.

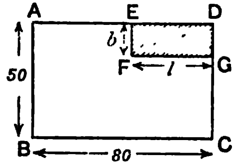


FIG. 8.

14. How many hours is a person in bed in the following cases ?

Time of going to bed	10 p.m.	9 p.m.	9 p.m.	y p.m.
Time of getting up	7 a.m.	7 a.m.		6 a.m.

Make a general statement.

15. Fig. 9 represents the floor of a room with right-angled corners. Find the lengths of DF, EF

(i) with the data in the figure ;

(ii) if AB $=a$ ft., BC $=c$ ft., AF $=14$ ft., CD $=12$ ft.

(iii) if AB $=a$ ft., BC $=c$ ft., AF $=x$ ft., CD $=y$ ft.

16. Fig. 10 represents a rectangular enclosure ; the dimensions

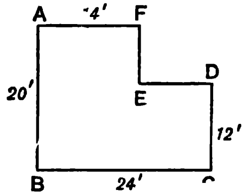


FIG. 9.

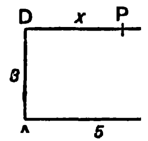


FIG. 10.

are given in feet. Write down the distance of B from P, measured round the enclosure, (i) through A and D, (ii) through C.

Find also these distances if AB $=b$ ft., AD $=d$ ft.

17. When a man is 40 years old, his son is 10 years old. Copy the given table, complete it, and make a general statement.

Age of Father	40	50			f	x
Age of Son	10		6	s		y

18. The area of the floor of a rectangular room is 180 sq. ft. Copy the given table, complete it, and make a general statement.

Length of room in feet -	18		l		x
Breadth of room in feet -		12		b	y

Notation in Algebra

The operation, "multiply the number N by 5," might be written $N \times 5$; to save time, it is written 5N.

Similarly, $N \times \frac{1}{2}$ which equals $\frac{1}{2} \times N$ is written $\frac{1}{2}N$ or $\frac{N}{2}$; and $N \times \frac{3}{4}$ is written $\frac{3}{4}N$ or $\frac{3N}{4}$.

Just as $5 \times 9 = 9 \times 5$, so $a \times b = b \times a$, and the short-hand form is ab or ba. But, just as $6 \times 1 = 1 \times 6 = 6$, so $N \times 1 = 1 \times N = N$; we therefore write N instead of 1N.

In Arithmetic, 57 means $5 \times 10 + 7$ and $5\frac{3}{4}$ means $5 + \frac{3}{4}$; but in Algebra 5N *always* means "five times N" or "N times five"; **ab always means a × b or b × a.**

In other respects, notation in Algebra is the same as in Arithmetic:

$5 \times 5 \times 5$ is written 5^3; $N \times N \times N$ is written N^3.

$2 \div 3$ is written $\frac{2}{3}$; $a \div b$ is written $\frac{a}{b}$.

$\sqrt{6}$ is the square root of 6; $\sqrt{y}$ is the square root of y.

$\frac{3}{7} + \frac{2}{7}$ is equal to $\frac{3+2}{7}$; $\frac{a}{x} + \frac{b}{x}$ is equal to $\frac{a+b}{x}$.

Example 2. What is the meaning of $3N - 7$? What is its value if N stands for 5?

3N means "multiply N by 3" or "multiply 3 by N."

To obtain the value of $3N - 7$, multiply N by 3 and then subtract 7 from the result.

If N stands for 5, $3N - 7 = 3 \times 5 - 7 = 15 - 7 = 8$.

Example 3. If r stands for 8, what is the value of $r + 2$

$$r = 8, \quad \therefore \frac{r+2}{r} = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4} = 1\frac{1}{4}.$$

EXERCISE I. c

State in words the meaning of the following, and *afterwards* state their values if $N=6$, $a=2$, $b=1$.

1. $3N$.	2. $N+3$.	3. $\frac{N}{3}$.	4. $N-3$.	5. N^2.
6. $2N+2$.	7. $\frac{1}{4}N$.	8. $\frac{6}{N}$.	9. $2N \div 3$.	10. $\frac{1}{2}N^2$
11. aN.	12. ab.	13. $a+b$.	14. $\frac{N}{a}$.	15. b^2.
16. bN.	17. $2aN$.	18. $2a+N$.	19. $N-2a$.	20. $a-2b$.
21. abN.	22. $3a^2$.	23. $N-b$.	24. $3+a$.	25. $5a-N$.
26. $\frac{N}{b}$.	27. $3+aN$.	28. $\frac{a+b}{N}$.	29. $3ab$.	30. $a+bN$.

If $x=3$, $y=5$, $c=8$, find the values of the following :

31. $2x$.	32. x^2.	33. $x+2$.	34. $x-2$.	35. $\frac{x}{2}$.
36. xy.	37. yx.	38. $2xy$.	39. $y-x$.	40. $2x-y$.
41. $c+x$.	42. cx.	43. $c-2x$.	44. $cy-10x$.	45. $2y^2$.

Write down the results of the following operations :

46. Add 2 to N.
47. Multiply a by 3.
48. Half of b.
49. Subtract 5 from c.
50. Four times l.
51. Add 1 to x.
52. Two-thirds of p.
53. Double y and add 2.
54. Three less than n.
55. Divide 12 by t.
56. Increase e by 5.
57. Halve z and then add 2.
58. Half the sum of s and 2.
59. Decrease k by 1.
60. Multiply N by 4 and subtract 2 from the result.
61. Divide r by 5 and subtract the result from 10.
62. Square c and multiply the result by 6.
63. Multiply together b, a, c and double the result.
64. Multiply r by s and divide the result by 5.

Like Terms

If an expression consists of various parts, some of which are connected by + or − signs, and others by × or ÷ signs, the parts connected by + or − signs are called *terms*. A formal discussion of like and unlike terms is best taken later, see p. 44.

Example 4. Generalise the statement, $9+9+9+9+9=9\times 5$.

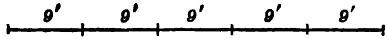


FIG. 11.

A straight fence, see Fig. 11, is made up of 5 hurdles, each 9 ft. long;

Its total length $=(9+9+9+9+9)$ ft. $=9$ ft. $\times 5=9\times 5$ ft.

Suppose each hurdle is x feet long;

The length of the fence $=(x+x+x+x+x)$ ft. $=x$ ft. $\times 5=5x$ ft.

Thus, $$\mathbf{x+x+x+x+x=x\times 5=5x.}$$

Example 5. Generalise the statement, $3\times 9+5\times 9=8\times 9$.

A fence is made by first setting up 3 hurdles, each 9 ft. long, and then adding 5 more hurdles, each 9 ft. long.

The lengths of the two portions are 9 ft. $\times 3$ and 9 ft. $\times 5$.

There are in all $(3+5)$ hurdles $=8$ hurdles;

$$\therefore \text{ total length} =(3\times 9+5\times 9) \text{ ft.} =8\times 9 \text{ ft.}$$

Suppose each hurdle is x feet long.

The lengths of the two portions are x ft. $\times 3$ and x ft. $\times 5$, which equal $3x$ ft. and $5x$ ft.; and the total length is x ft. $\times 8$ or $8x$ ft.

$$\therefore \text{ total length} =(3x+5x) \text{ ft.} =8x \text{ ft};$$

Thus, $$\mathbf{3x+5x=8x.}$$

This is simply a short-hand statement.

$$3x=x+x+x \text{ and } 5x=x+x+x+x+x.$$

$$\therefore\ 3x+5x=x+x+x+x+x+x+x+x=8x.$$

Example 6. Simplify $7x-2x$.

Suppose a fence is made of 7 hurdles, each x feet long; its total length is $7x$ feet.

Now remove two of the hurdles; then the length of the part removed is $2x$ feet.

But 5 hurdles remain; $\therefore$ the length of the remainder is $5x$ feet;

$$\therefore\ \mathbf{7x-2x=5x.}$$

This is a short-hand statement for the following:

$$x+x+x+x+x+x+x-x-x=x+x+x+x+x=5x.$$

Example 7. Write more shortly:

(i) $5a\times 3$; (ii) $x\times 3y$; (iii) $2a\times 5b$;

(iv) $2a\times 5ab$; (v) $6a\div 2$; (vi) $\frac{5a}{6}\times 4$.

(i) $5a \times 3 \quad = 5a + 5a + 5a = \mathbf{15a}.$

(ii) $x \times 3y \quad = x \times 3 \times y = 3 \times x \times y = \mathbf{3xy}.$

(iii) $2a \times 5b \quad = 2 \times a \times 5 \times b = 2 \times 5 \times a \times b = \mathbf{10ab}.$

(iv) $2a \times 5ab = 2 \times a \times 5 \times a \times b = 2 \times 5 \times a \times a \times b = \mathbf{10a^2b}.$

(v) $6a \div 2 \quad = \dfrac{6 \times a}{2} = 3 \times a = \mathbf{3a}.$

(vi) $\dfrac{5a}{6} \times 4 \quad = \dfrac{5 \times a \times 4}{6} = \dfrac{5 \times 2 \times a}{3} = \dfrac{\mathbf{10a}}{\mathbf{3}}.$

To simplify an expression containing several terms, *work from the left* unless, just as in Arithmetic, brackets or multiplication and division signs show that operations must be performed in a different order. Thus $11 - 3 + 2 = 8 + 2 = 10$; similarly,

$$11x - 3x + 2x = 8x + 2x = 10x, \text{ and } 7y + 2y - 5y = 9y - 5y = 4y.$$

EXERCISE I. d

Write down short-hand forms for the following:

1. $A \times 3$. 2. $B \times 1$. 3. $C \times C$. 4. $n + n$.

5. $x + x + x$. 6. $y + y + y + y$. 7. $3z + 3z$. 8. $3a \times 2$.

9. $3a \times 5$. 10. $3b + 2b$. 11. $4c + c$. 12. $4d \times 2$.

13. Three times N. 14. Three times $2p$. 15. Four times $3q$.

16. Half of A. 17. One-quarter of 4B. 18. Two-thirds of C.

19. $12e \div 3$. 20. $\dfrac{15f}{5}$. 21. $\dfrac{t}{2} \times 6$. 22. $\dfrac{3t}{2} \times 6$.

23. $4 \times 7y$. 24. $4 \times \dfrac{y}{2}$. 25. $2 \times \dfrac{3z}{4}$. 26. $5x \div 6$.

27. $8x \div 6$. 28. $\dfrac{3y}{4} \times 8$. 29. $6z \times \frac{1}{2}$. 30. $6a \times \frac{3}{4}$.

31. $3x \div 12$. 32. $8r \div 12$. 33. $s \div 1$. 34. $t \div \frac{1}{2}$.

35. $4y \times \frac{3}{4}$. 36. $4z \div \frac{2}{3}$. 37. $6a \div \frac{1}{4}$. 38. $10b \div \frac{2}{5}$.

39. $3 \times 7 + 2 \times 7$. 40. $3t + 2t$. 41. $4p + 4p$.

42. $5 \times 8 + 8$. 43. $5y + y$. 44. $z + 3z$.

45. $6 \times 9 - 2 \times 9$. 46. $6x - 2x$. 47. $3f - f$.

48. $4t - 3t$. 49. $5p - 5p$. 50. $2q - q$.

51. $a + a + 2a$. 52. $3b + 2b + b$. 53. $4c + c + 4c$.

54. $3x + 0 + x$. 55. $4y + y - 3y$. 56. $2z + 5z - 4z$.

57. $8r - 2r - 5r$. 58. $4s - 0 - s$. 59. $6t - 2t + 3t$.

60. $c \times 3c$. 61. $2d \times 3d$. 62. $a \times 3b$. 63. $2x \times 4y$.

64. $3 \times 3z$. 65. $5 \times 4a$. 66. $bc \times b$. 67. $bc \times 2b$.

68. $3bc \times 3b$. **69.** $3bc \times 3bc$. **70.** $2x \times 3xy$. **71.** $3x^2 \times 2$.

72. $\frac{3a}{2} \times 4a$. **73.** $\frac{b}{3} \times \frac{b}{3}$. **74.** $\frac{a}{5} \times 5a$. **75.** $\frac{a}{2} \times 6b$.

Numbers and Quantities

Letters in Algebra are used to represent numbers, not numbers-of-things.

A letter may stand for 2, 15, $\frac{3}{4}$, etc., but *not* for 2 pence, 15 days, $\frac{3}{4}$ mile, etc.

A number-of-things is called a **quantity.**

When dealing with quantities, **always state what the unit is,** as in the following examples:

A parcel weighs W lb.; a book costs C shillings; a room is h feet high; a tank holds n gallons.

EXERCISE I. e

1. I walk s miles and then ride 4 miles. How far do I go? What is the answer if $s = 3$?

2. A box weighs 3 lb.: I put into it two parcels, one weighing 2 lb., the other W lb. What is the total weight? What is the answer if W = 5?

3. A milkman sells m gallons of milk out of 20 gallons; how much has he left? What is the answer if $m = 16$?

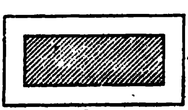
4. A bus fare is 4 pence; what is the cost of (i) 5 journeys, (ii) n journeys?

5. I buy c apples and then buy 2 more; how many altogether? What is the answer if $c = 10$?

6. I buy a bunch of $3t$ bananas and cut off 3 of them; how many are left? What is the answer if $t = 5$?

7. A basket, weighing W lb. when empty, contains $6n$ lb. of apples; if $2n$ lb. of apples are sold, what is the weight of the basket and the remaining apples? What is the answer if W = 4 and $n = 5$?

8. Fig. 12 represents a carpet on the floor of a room; there is a margin 2 feet wide all the way round. What is the length and breadth of the carpet? What is the answer if $l = 30$, $w = 15$?



w ft.

l ft.

FIG. 12.

9. I can walk 4 miles an hour. How far can I walk in 3 hours, t hours, $3t$ hours?

10. What is the cost of the following ?

(i) 5 lb. of butter at l pence per lb.

(ii) 2 oz. of pepper at x pence per oz.

(iii) m yd. of silk at 12 shillings per yd.

(iv) x gallons of oil at p shillings per gallon.

(v) R feet of pipe at n pence per foot.

(vi) N dozen note-books at P pence each book.

11. How many inches are there in 4 ft., 4 ft. 9 in., l ft., l ft. m in., v yd. ?

12. How many shillings have I left if I spend
(i) 5s. out of £2 ; (ii) 5s. out of £N ; (iii) Ps. out of £Q ?

13. What is the total length in inches of the wire used to make the grid in Fig. 13 ?
Each inch of wire weighs $\frac{1}{8}$ oz. ; what is the weight of the grid ?
What is the value of each answer if $a = 2$, $b = 8$?

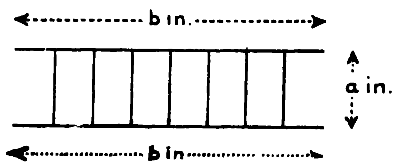


Fig. 13.

14. It is now a quarter past ten ; in how many minutes will it be (i) 10.25, (ii) t minutes past ten, (iii) n minutes to 11 ?

15. A man buys a horse for £40 and sells it for £P ; what is his profit ?

16. By selling a watch for z shillings, I gain 10 shillings ; how much did it cost me ?

17. (i) v apples are shared equally among 5 boys ; how many does each have ?
(ii) v apples are shared equally among 5 boys and x girls ; how many does each have ?

18. A car uses a gallon of petrol every 24 miles. (i) How far will the car run on $2\frac{1}{2}$ gall., N gall. ? (ii) How much petrol is used to run 1 mile, 10 miles, s miles ?

19. Fig. 14 represents a short flight of steps from A to B. Each step is h inches high and d inches deep. What length of carpet is needed to run from A to B ?

Fig. 14.

20. A jug holds 2 pints. How many jugs are needed for k pints, n gallons ?

21. A pail holds P pints. How many times can it be filled from a tank containing 100 pints, N pints, V gallons ?

22. A railway porter is paid b shillings a week and receives in tips n shillings a day. How much does he get each week, working 6 days ?

23. A family uses 3 loaves of bread a day. How many loaves are needed for n days, t weeks? How long will x loaves last?

24. Eggs are 3 pence each. How much change (in pence) is there out of half a crown, if you buy z eggs?

25. How long (in hours) does it take to travel 20 miles at 10 m.p.h., v m.p.h.? How long for s miles at u m.p.h.?

26. A boy is now y years old and his father is now three times as old. How old will each be in 3 years' time?

27. A row of houses is l yards long; each house is w yards wide. How many houses are there in the row?

28. A bookshelf is l feet long; how many books, each t inches thick, will it hold?

29. Take the number n; square it and multiply the result by 6.

30. Take the number p; multiply it by 4 and subtract 4.

[*Note. For additional examples, see Appendix, Ex. S.* 1, p. 274.]

Meaning of Brackets

It was pointed out on p. 1 that the contents of a bracket may be regarded as equivalent to a single number.

Thus, $(7+3)$ means the number obtained by adding 3 to 7; and $(\mathrm{N}+5)$ means the number obtained by adding 5 to N.

The product of 9 and $(7+3)$ is written $9\,(7+3)$.

The product of 9 and $(\mathrm{N}+5)$ is written $9\,(\mathrm{N}+5)$.

Similarly $(p-q)\div 7$ means "subtract q from p and divide the result by 7." It is usually written $\frac{p-q}{7}$ or $\frac{1}{7}(p-q)$.

Also, just as a^2 means $a\times a$, so $(x+y)^2$ means "add y to x and multiply the result by itself."

Brackets show the order in which operations must be performed.

Thus $5+2(3+4)$ means "add 4 to 3, double the sum, add the result to 5."

$$5+2(3+4)=5+2\times 7=5+14=19.$$

But $(5+2)3+4$ means "add 2 to 5, multiply the sum by 3, add 4 to the result."

$$(5+2)3+4=7\times 3+4=21+4=25.$$

And $(5+2)(3+4)$ means "add 2 to 5, add 4 to 3, multiply the first sum by the second sum."

$$(5+2)(3+4)=7\times 7=49.$$

Example 8. 8 oz. of cocoa are packed in a tin which weighs t oz. when empty. What is the weight of 5 tins of cocoa ?

One tin when full of cocoa weighs $(t+8)$ oz.

$$\therefore \text{ 5 full tins weigh } 5(t+8) \text{ oz.}$$

Again, the 5 empty tins weigh $5t$ oz., and the cocoa by itself weighs 5×8 oz., $\therefore$ the whole weighs $(5t+5\times 8)$ oz.

$$\therefore\ 5(t+8) \text{ oz.} = (5t+5\times 8) \text{ oz.}$$

$$\therefore\ 5(t+8) = 5t+5\times 8.$$

Thus, if an expression in a bracket is multiplied by a number, each term in the bracket must be multiplied by that number, when the bracket is removed.

Test this statement numerically. Suppose that the tin weighs 2 oz.; then $t=2$.

$$5(t+8) \quad = 5(2+8) \qquad = 5\times 10 \ = 50.$$

and
$$5t+5\times 8 = 5\times 2+5\times 8 = 10+40 = 50.$$

$$\therefore \text{ if } t=2, \text{ each expression equals } 50.$$

EXERCISE I. f

State in words the meaning of the following; and *afterwards* find their values if $N=6$, $a=2$.

1. $2(N+1)$. **2.** $(N+3)a$. **3.** $5(N-5)$. **4.** $3(a+3)$.

5. $a(3N+2)$. **6.** $(2a-1)4$. **7.** $N(a+1)$. **8.** $(2a-1)N$.

Find the values of the following, if $x=3$, $y=5$, $c=8$.

9. $2(c-x)$. **10.** $c(y-x)$. **11.** $(x-2)y$. **12.** $\frac{x+y}{c}$.

Remove the brackets in the following; and *afterwards* test the results for the given values.

13. $3(2l+1)$; $l=4$. **14.** $4(a-3)$; $a=5$.

15. $2(3p-4)$; $p=3$. **16.** $3(5+2x)$; $x=2$.

17. $a(b+3)$; $a=4$, $b=2$. **18.** $2p(3a-4)$; $p=5$, $a=2$.

19. $3x(y+z)$; $x=2$, $y=3$, $z=4$.

20. $5a(b-2c)$; $a=4$, $b=5$, $c=2$.

Find the values of the following, if $a=3$, $b=2$, $c=6$, $d=5$.

21. $a+b(c+d)$. **22.** $(a+b)(c+d)$. **23.** $(a+b)c+d$.

24. $a+(b+c)d$. **25.** $a+bc+d$. **26.** $(b+c)\div a$.

27. $b+c\div a+d$. **28.** $(b+c)\div(a+d)$. **29.** $(b+c)\div a+d$.

30. $c\div(a+d)$. **31.** $b+c\div(a+d)$. **32.** $b\div a+d\div c$.

33. $(a+b)^2$. **34.** $a^2+(b+c)^2$. **35.** $(a+b+c)^2$.

Use of Brackets

Example 9. 8 oz. of cocoa are packed in a tin which weighs t oz. when empty; 30 tins of cocoa are packed in a box weighing W oz. What is the total weight?

One tin, when full of cocoa, weighs $(8+t)$ oz.

$\therefore$ 30 tins of cocoa weigh $30(8+t)$ oz.

$\therefore$ the case of cocoa weighs $\mathbf{W}$ **oz.** $+ \mathbf{30(8+t)}$ **oz.**

It is simpler to write this answer in the form $[\mathbf{W}+\mathbf{30(8+t)}]$ **oz.**

Note. If one set of brackets is enclosed, as here, inside another set, different shapes of brackets should be used, because this makes it easier to see what the expression means.

EXERCISE I. g

Use brackets when answering the following questions; do **not** *remove the brackets.*

1. 10 lb. of jam fills a jar which weighs w lb. when empty. What is the weight of 10 full jars? The jars are packed in a box weighing P lb. What is the total weight?

2. I have 10 coins; n of them are sixpennies and the rest are shillings. What is their value in pence?

3. A cellar contains k bottles; six dozen of them hold a pint each and the rest a quart each. How many pints are there altogether?

4. A man buys glasses at the rate of 6d. each for the first dozen and 5d. each for the rest. What is the cost in pence (i) of 20 glasses, (ii) of x glasses, where $x > 12$?

5. In Fig. 15, AB is 10 inches, AP is l inches long. PB is divided into three equal parts. What is the length of each part?

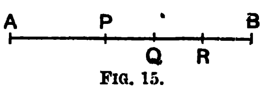


FIG. 15.

6. In Fig. 15, AB is s inches, AP is 2 inches; and PB is divided at Q, R into three equal parts. What are the lengths of (i) PQ; (ii) AQ; (iii) BR; (iv) AR?

7. A man at a hotel is charged £1 a day for the first four days, and 16s. a day afterwards. What is the amount of his bill in shillings for (i) 7 days, (ii) n days, if $n > 4$?

8. A journey by car takes $3\frac{1}{2}$ hours; for the first t hours, the speed is 20 miles per hour, and for the remainder it is 15 miles per hour. What is the distance travelled $(t < 3\frac{1}{2})$?

9. A man's rate of pay is as follows: ordinary time, 10d. an hour; overtime, 15d. an hour. The regular working day is 7 hours. What does he receive for a day on which he works (i) 5 hours; (ii) 10 hours; (iii) t hours if $t<7$; (iv) T hours if $T>7$? Answer in pence.

10. A workman is paid xs. yd. per day. What does he receive (in shillings) for t days' work?

Write down expressions for the following, Nos. 11-24:

11. The result of multiplying $a-b$ by $2\frac{1}{2}$.

12. The result of subtracting $P+2Q$ from R.

13. The result of subtracting $p+q$ from $r-s$.

14. Five times the sum of x and y.

15. Subtract c from d and divide the result by 3.

16. Three-quarters of the sum of A and B.

17. The average of p, q, r.

18. The product of $2a$ and $(y-z)$.

19. The number by which 10 exceeds the sum of x and t.

20. The square of the sum of a and b.

21. The number of pence in $(p+q)$ shillings.

22. The number of feet in $(a+b)$ yards l inches.

23. Subtract from a half the sum of b and c.

24. The product of three consecutive whole numbers of which (i) l is the least; (ii) g is the greatest; (iii) m is the middle number.

25. A box full of sugar weighs W lb. and when empty weighs w lb. What is the weight of the sugar in n boxes?

26. When a boy is x years old his father is y years old; how much younger is the boy than his father? How old is the boy when his father is z years old?

27. A cask when empty weighs P lb. and when full weighs Q lb. What is the weight of the contents when the cask is full? What is the weight of the cask and contents when the cask is half-full?

28. A man was earning $(a+3b)$ shillings a week. What are his new wages if the old wages are increased by one-tenth?

29. The letter rate for inland postage is as follows: $1\frac{1}{2}$d. for the first 2 oz. and $\frac{1}{2}$d. for each additional 2 oz. or part thereof. What is the cost in pence of sending a letter weighing W oz., if W is an even whole number?

30. Use the rule in No. 29 to find the cost in pence of postage on a letter weighing $(2W+1)$ oz. where W is a whole number.

CHAPTER II

FORMULAE

Construction of Formulae

THE following illustrative examples are intended for oral discussion.

Example 1. The Area of a Rectangle.

If you have a rectangle 4 in. long, 3 in. broad, you can divide it as in Fig. 16, so that there are 3 rows and each row contains 4 one-inch squares. Therefore there are 4×3 one-inch squares altogether. But the area of a one-inch square is called 1 sq. inch.

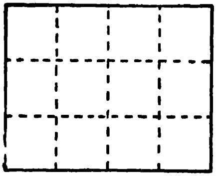


FIG. 16.

$\therefore$ the area of the rectangle is 4×3 sq. in.

It would be a waste of time to use this process whenever the area of a rectangle is required. We therefore look for *the general method.*

Suppose a rectangle is l inches long and b inches broad. Show, by repeating the argument used above, that, if l and b are whole numbers, its area is $l \times b$ sq. inches. Call the area A sq. inches.

FIG. 17.

$$\text{Then } \mathbf{A = l \times b}.$$

This relation is called a formula. Although only proved here for integral values of l and b, it can be shown to be true for all values; we therefore use it whenever we wish to find the area of a rectangle.

Example 2. The Measurement of Speed.

A policeman times a car over a measured distance of 270 feet.

If the car takes 6 seconds, its average speed is $\frac{270}{6}$ ft. per sec. $= 45$ ft. per sec.

If the car takes $4\frac{1}{2}$ seconds, its average speed is $\frac{270}{4\frac{1}{2}}$ ft. per sec. $= 60$ ft. per sec.

The policeman wishes to know the speed in miles per hour.

$$45 \text{ ft. per sec.} = \frac{45 \times 60 \times 60}{3 \times 1760} \text{ miles per hour}$$

$$= \frac{45 \times 15}{22} \text{ m.p.h.} = 30\tfrac{15}{22} \text{ m.p.h.}$$

Similarly, 60 ft. per sec. may be expressed in miles per hour; but time is saved by using the general formula.

If the car takes t sec. to travel s ft., its average speed is $\frac{s}{t}$ ft. per sec.

$$= \frac{s}{t} \times \frac{60 \times 60}{3 \times 1760} \text{ miles per hour}$$

$$= \frac{s}{t} \times \frac{15}{22} \text{ m.p.h.} = \frac{15s}{22t} \text{ m.p.h.}$$

Therefore the average speed of a car, which takes t sec. to travel s ft., is V miles per hour, where

$$V = \frac{15s}{22t}.$$

This is the policeman's formula; he does not waste time by working through the argument by which it is proved, whenever he uses it.

EXERCISE II. a

[If brackets occur in the Answer, do not remove them.]

1. What is the cost in pence of an inland telegram with (i) 10 words, (ii) 17 words, (iii) n words? [The charge is: 12 words or less, 1s.; each word more, 1d.]

2. Find a formula for the third angle of a triangle, given the other two angles. A°, B°. [Invent an example with special numbers.]

3. Find a formula for the time a train takes to go a given distance, s miles, at v miles an hour. [Invent an example with special numbers.]

4. When making tea, put in one spoonful for each person and one for the pot. How much tea is required for n people (i) if one teapot is used, (ii) if k teapots are used?

5. Find a formula for the cost of n collars, sold as follows:

Number of collars -	1	4	12
Total cost - -	1s. 3d.	5s.	15s.

6. A member of a club receives 3 free tickets and is charged 5s. for each additional ticket. How much does he pay for (i) 7 tickets, (ii) p tickets? Give each answer firstly in shillings, secondly in £.

7. String is sold by weight, as follows:

Weight - - -	8 oz.	12 oz.	1 lb.
Total cost - -	1s. 4d.	2s.	2s. 8d.

Find a formula for the cost of W oz.

8. Use the data of No. 7 to find a formula for the weight of a ball of string which costs (i) p pence, (ii) z shillings, (iii) x shillings y pence.

9. A book is t in. thick, each cover is c in. thick, and there are n sheets. What is the thickness of each sheet? [First invent a numerical example.]

10. A piece of cotton is wound n times round a cylinder. When unwound, it measures l inches. What is the girth of the cylinder (i) in inches, (ii) in feet?

11. The wheel of a car makes r revolutions when the car travels s yards. What is the circumference of the wheel (i) in yd. (ii) in ft.?

12. What is (i) the perimeter, (ii) the area of the rectangle in Fig. 18?

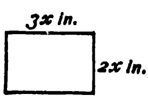


FIG. 18.

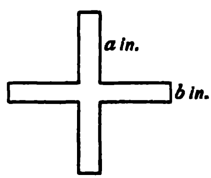


FIG. 19.

13. Fig. 19 represents a cross with four equal arms. What is (i) the perimeter, (ii) the area of the cross?

14. How many times can a jug holding p pints be filled from a cask holding g gallons? [1 gallon = 8 pints.]

15. In Fig. 20, AB is longer than BC by the same amount that BC is longer than CA. What is the distance of A from B?

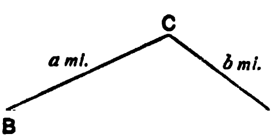


FIG. 20.

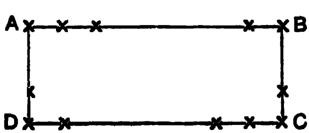


FIG. 21.

16. How many posts are required for a straight fence AB, if a post is needed every 5 yards, see Fig. 21, and if the length of the fence is 35 yards, 100 yards, $5l$ yards, l being an integer?

17. How many posts are required for the fence of a field ABCD, see Fig. 21, $5l$ yards long, $5b$ yards broad, if a post is needed every 5 yards; l and b being integers?

18. A closed box, see Fig. 22, is x in. long, x in. wide, y in. deep, outside measurements. Find (i) the total length of all its edges, (ii) the total area of its outside surface, (iii) its volume.

What do these results become in terms of x, if $y = 2x$?

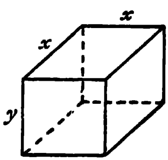


FIG. 22.

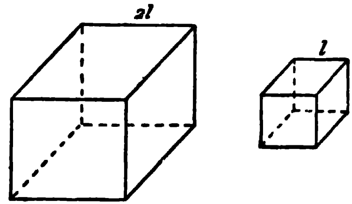


FIG. 23.

19. Fig. 23 represents two cubical tins, one of side $2l$ in., the other of side l in., inside measurements; the larger is full of water, the smaller is empty. How much water is left in the larger tin, when the smaller has been filled from it ? How many more tins, each equal to the smaller tin, can be filled from the larger ?

20. Neither tin in Fig. 23 has a lid. With the data of No. 19, find the area of the inside surface of each tin.

21. Find the area of Fig. 24 by taking the sum of two rectangles.

22. Find the area of Fig. 25 by taking the difference of two rectangles.

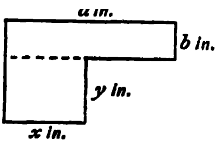


FIG. 24.

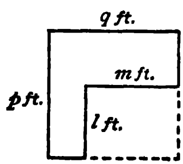


FIG. 25.

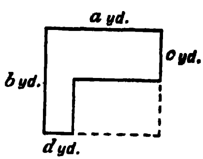


FIG. 26.

23. Find the area of Fig. 26 by taking the difference of two rectangles.

24. Find the perimeter of the areas bounded by the continuous lines in (i) Fig. 24, (ii) Fig. 25, (iii) Fig. 26.

25. In Fig. 27, AB $= b$ in., AC $= c$ in., and P is the mid-point of BC. What is the length of (i) BC, (ii) PC, (iii) AP ?

FIG. 27.

26. Fig. 28 represents a rectangular brick wall pierced by three equal windows, each h ft. high, w ft. wide. The units for the dimensions shown in the figure are feet. What is (i) the length and height of the wall, (ii) the area of the brickwork ?

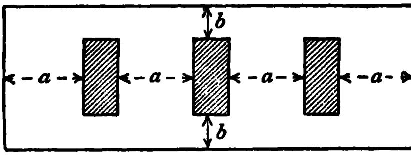


FIG. 28.

27. Fig. 29 represents a sheet of paper from which equal squares, side h in., have been cut away at each corner. The paper is folded to form a box, without a lid ; the dotted lines show the creases. (i) Find the length, breadth and height of the box. (ii) What is the volume of the box ? (iii) Write down, *without any working*, the total area of its outside surface.

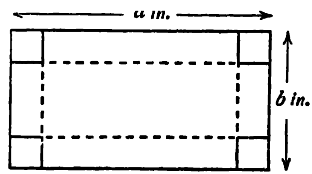


FIG. 29.

[*Note. For additional examples, see Appendix, Ex. S.* 2, p. 275.]

Substitution

The process of finding the numerical value of an expression, when the letters it contains stand for given numbers, is called **substitution.** It is required for making use of formulae.

Example 3. If $x=3$, $y=2$, find the values of :

(i) $2x^2$; (ii) $5xy$; (iii) xy^3.

(i) $2x^2=2\times x^2=2\times x\times x$.

$\therefore$ if $x=3$, $2x^2=2\times 3\times 3=18$.

(ii) If $x=3$, $y=2$, $5xy=5\times 3\times 2=30$.

(iii) If $x=3$, $y=2$, $xy^3=3\times 2\times 2\times 2=24$.

EXERCISE II. b

If $x=4$, $y=3$, find the values of the following :

1. $2x+y$. **2.** $2xy$. **3.** y^3. **4.** $2x^2$.

5. $xy-x$. **6.** $2y$ **7.** x^2-y^2. **8.** $(x-y)^2$.

9. $\frac{1}{2}xy^2$. 10. $3x - 4y$. 11. $x - \dfrac{6}{y}$. 12. $6 + \dfrac{x}{2}$.

If $p = 3$, $q = 0$, find the values of the following :

13. $q + 2$. 14. $2q$. 15. $2pq$. 16. $3p - 2q$.

17. p^2q. 18. $p^2 + q^2$. 19. $p(p - q)$. 20. $\dfrac{q}{p}$.

If $r = 3$, $s = 5$, $t = 4$, find the values of the following :

21. rst. 22. $2rst$. 23. $r + s + t$. 24. $t(r + s)$.

25. $r + s - 2t$. 26. ts^2. 27. $\dfrac{rt}{s}$. 28. $s^2 - 2rt$.

If $a = 1$, $b = \frac{1}{2}$, find the values of the following :

29. a^2. 30. b^3. 31. ab. 32. $a + b$.

33. $6ab$. 34. $1 - b$. 35. $\dfrac{1}{b}$. 36. a^2b^2.

If $c = 3$, find the values of the following :

37. $c^2 - 1$. 38. $2c^3$. 39. $\dfrac{c}{12}$. 40. $\dfrac{24}{c}$.

41. $c^2 - 3c$. 42. $(c - 1)^2$. 43. $(2c)^2$. 44. $c(c + 2)$.

If $e = 6, f = 2$, $g = 0$, find the values of the following :

45. $ef + fg$. 46. $3f^2 + 2eg$. 47. $\dfrac{e + f}{e - f}$.

48. $\frac{3}{4}efg$. 49. $e^2 - 3f^2$. 50. $(e - f)^2 + (f - g)^2$.

If $l = \frac{1}{2}$, $m = \frac{1}{3}$, find the values of the following :

51. $l - m$. 52. lm. 53. $\dfrac{l}{m}$. 54. $1 - l$.

55. $\dfrac{1}{l}$. 56. $6lm^2$. 57. $2l^2 + 3m^2$. 58. $\dfrac{3}{l} - \dfrac{2}{m}$.

59. If $a = 3b$ and $b = 4$, what is a ?
60. If $y = x^2$ and $x = 5$, what is y ?
61. If $R = \frac{1}{2}r$ and $r = 16$, what is R ?
62. If $y = 2x + 3$ and $x = 4$, what is y ?
63. If $c + d = 8$ and $d = 3$, what is c ?
64. If $pv = 48$ and $v = 8$, what is p ?
65. If $s = 3t$ and $t = 2$, what is st ?
66. If $N - n = 3$ and $n = 7$, what is (i) N ; (ii) $N + n$; (iii) Nn ?
67. If $y - z = 2$ and $y = 9$, what is (i) z ; (ii) $y + z$?

68. If $\frac{a}{b}=6$ and $b=2$, what is (i) a ; (ii) ab ?

69. If $bl=3$ and $l=2$, what is (i) b ; (ii) $2(b+l)$?

70. If $y=\frac{1}{2}x-1$ and $x=6$, what is y ?

71. If $y=x^2-2x$ and $x=5$, what is y ?

[*Note.* *For additional drill-examples, see Exercise E.P.* 1, p. 133.]

Use of Formulae

Example 4. From a masthead, h feet above the surface of the sea, it is possible to see a distance of $\sqrt{\left(\frac{3h}{2}\right)}$ miles. How far can an observer see, if he is (i) at the top of a mast 54 feet above the sea, (ii) at the top of a cliff 150 feet high ?

(i) Put $h=54$. Then the distance of the horizon is

$$\sqrt{\left(\frac{3\times 54}{2}\right)} \text{ mi.} = \sqrt{(81)} \text{ mi.} = \textbf{9 miles.}$$

(ii) Put $h=150$. Then the distance of the horizon is

$$\sqrt{\left(\frac{3\times 150}{2}\right)} \text{ mi.} = \sqrt{(225)} \text{ mi.} = \textbf{15 miles.}$$

The process of obtaining special results from a general formula is called **substituting in the formula.**

EXERCISE II. c

1. M miles is nearly the same distance as $\frac{8\text{M}}{5}$ kilometres. Find the number of kilometres in 20 miles.

2. P pints of water weigh about $1\frac{1}{4}$P lb. What is the weight of (i) 2 pints of water, (ii) 1 gallon of water ? Find in ounces the weight of half a pint of water.

3. v miles an hour is the same speed as $\frac{22v}{15}$ feet per second. Express in ft. per sec. (i) 60 m.p.h. ; (ii) 45 m.p.h. ; (iii) 5 m.p.h.

4. s yards a minute is the same speed as $\frac{3s}{88}$ miles an hour. Express in m.p.h. (i) 880 yd. per min., (ii) 88 yd. per min.

5. When making tea for N people, if t teapots are required, you must use $(\text{N}+t)$ teaspoonfuls of tea. How much is used for (i) 5 people (1 teapot) ; (ii) 15 people (2 teapots) ; (iii) 30 people (3 teapots) ?

6. With summer time, the middle of the day may be taken as 1 o'clock ; and so, if the Sun rises at t o'clock a.m., it sets at $(14-t)$ o'clock p.m. On May 20, in London, the Sun rises at 5 a.m., when does it set ? What is the length of the day ?

7. Fig. 30 represents a polygon with 5 sides (*i.e.* a pentagon). If a polygon has n sides, the sum of its interior angles is $(2n-4)$ right angles. What is the sum of the angles of (i) a quadrilateral, (ii) a pentagon, (iii) a decagon (10 sides)? Does the formula hold for a triangle?

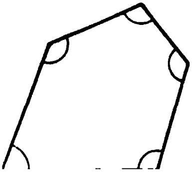


FIG. 30.

8. n postcards cost $(n+1)$ pence if $n<12$ and cost $(n+2)$ pence if $11<n<23$. What is the cost of (i) 8 postcards, (ii) 20 postcards?

9. For corrugated iron roofing, see Fig. 31, the relation between the pitch, P in., and the depth, d in., is $d=\frac{1}{4}P$.

The pitch should not be less than 3 inches; what can you say about the depth?

The pitch should not be more than 5 inches; what can you say about the depth?

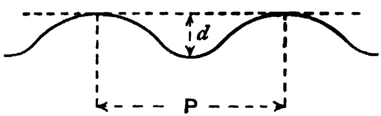


FIG. 31.

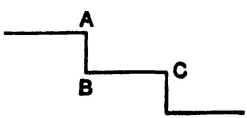


FIG. 32.

10. The "rise" AB, R inches, and the "tread" BC, T inches, of a staircase, see Fig. 32, are often connected by the rule $R=\frac{1}{2}(24-T)$. The tread should not be less than 9 inches; what can you say about the rise? The tread should not be more than 12 inches; what can you say about the rise?

11. Repeat No. 10, using the rule, $R=\frac{66}{T}$, which is sometimes employed.

12. The nth day of March is the same day of the week as the $(n+5)$th day of September. If the 1st of March is a Monday, what is the date of the first Monday in September? What day of the week is Sept. 1?

13. Using the rule in No. 12, if the 4th of March is a Monday, what is the date of the first Monday in September?

If the 27th of March is a Wednesday, what is the date of the last Wednesday in September?

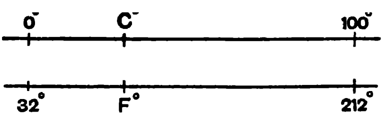


FIG. 33.

14. F° Fahrenheit is the same temperature as C° Centigrade, if $\frac{F-32}{212-32}=\frac{C}{100}$, see Fig. 33; this is equivalent to $F=32+\frac{9C}{5}$.

Express in degrees Fahrenheit; (i) 100° C., (ii) 0° C., (iii) 15° C., (iv) 35° C.

15. The formula connecting F and C in No. 14 may also be stated in either of the following ways:

$$\text{(i) } F = \tfrac{9}{5}(C + 40) - 40; \quad \text{(ii) } C = \tfrac{5}{9}(F + 40) - 40.$$

From (i), find F for 15° Centigrade; then use this value of F to find C from (ii).

Show in a similar way that the formulae agree for 50° Fahrenheit.

16. Use the rules in No. 15 to state *in words* how to convert degrees Centigrade to degrees Fahrenheit, and vice versa.

17. With the data of Fig. 34, the triangle is right-angled. What results are obtained by taking (i) $x=2$, (ii) $x=3$, (iii) $x=4$?

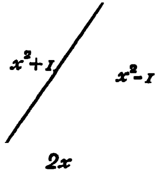


FIG. 34.

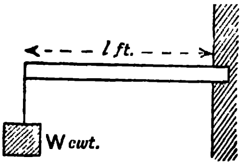


FIG. 35.

18. A parcel whose weight does not exceed $(3n - 1)$ lb. may be sent by post for $(3n+3)$ pence, where n is an integer. But parcels for which $n > 4$ are not accepted by the post office. State in words what this rule means by taking $n=1$, $n=2$, $n=3$, $n=4$.

19. An oak beam, l feet long, b inches wide, d inches thick, is built into a wall at one end and carries a load of W cwt. at the other end, see Fig. 35. It will break if $W > \frac{5bd^2}{4l}$. Will such a beam, 5 ft. long, 8 in. wide, 3 in. thick, break under a load of 1 ton?

If the thickness is 6 inches, instead of 3 inches, will it break under a load of $3\frac{1}{2}$ tons?

20. The marks obtained in an examination run from 20 to 70. They are converted so that an original mark n becomes t, where $t = 2(n - 20)$. What is the new mark if the original mark is (i) 31, (ii) 58, (iii) 40?

What is the new top mark and the new bottom mark?

21. A wind blowing at v miles an hour exerts a direct pressure of P lb. per sq. foot of surface it strikes, where $P = \frac{v^2}{200}$. What pressure must a hoarding 10 ft. high, 20 ft. wide, be able to withstand against (i) a breeze of 10 m.p.h., (ii) a gale of 25 m.p.h., (iii) a storm of 50 m.p.h.?

[*Note. For additional examples, see Appendix, Ex. S.* 3, p. 280.]

Generalised Statements about Numbers

Numbers represented by letters need not be whole numbers.

Example 5. Give a general statement which includes the following special facts :

$$4 \times 7 = 7 \times 4 \; ; \; 8 \times 6 = 6 \times 8 \; ; \; 3\tfrac{1}{2} \times 2\tfrac{3}{4} = 2\tfrac{3}{4} \times 3\tfrac{1}{2}.$$

In words, if we take any two numbers, the result of multiplying the first by the second is equal to that obtained by multiplying the second by the first.

Using letters, if a and b are any two numbers,

$$a \times b = b \times a.$$

Example 6. Give a general statement to include the following : and test one of them.

$$7^2 - 5^2 = 4 \times 6 \; ; \; 10^2 - 8^2 = 4 \times 9 \; ; \; 14^2 - 12^2 = 4 \times 13.$$

Test the last statement ; $14^2 = 196$, $12^2 = 144$;

$$\therefore \; 14^2 - 12^2 = 196 - 144 = 52 = 4 \times 13.$$

In words, if we take three consecutive integers, the square of the largest minus the square of the smallest equals four times the middle number.

Using letters, if l, $l+1$, $l+2$ are three consecutive integers,

$$(l+2)^2 - l^2 = 4(l+1).$$

The similarity of 3 or 4 special facts suggests that a general statement, which includes all of them, is true ; but you cannot be certain that this is so, unless you prove it. Try to prove the statement in Example 6 by using Fig. 65, p. 49, and stating the area of the whole figure and the area of each compartment.

EXERCISE II. d.

Give general statements which include the following :

1. $5+9=9+5$; $2+8=8+2$; $11+3=3+11$; $\mathsf{A}+\mathsf{B}= \ldots$.
2. $5+5=$ twice 5 ; $8+8=$ twice 8 ; $2\tfrac{1}{2}+2\tfrac{1}{2}=$ twice $2\tfrac{1}{2}$; $\mathsf{N}+\mathsf{N}= \ldots$.
3. $\tfrac{1}{3} \times 3 = 1$; $\tfrac{1}{7} \times 7 = 1$; $\tfrac{1}{10} \times 10 = 1$; $\dfrac{1}{c} \times c = \ldots$.
4. $5 \times \tfrac{4}{5} = 4$; $7 \times \tfrac{4}{7} = 4$; $11 \times \tfrac{4}{11} = 4$.
5. $6 \times 1 = 6$; $7 \times 1 = 7$; $12 \times 1 = 12$.
6. $3 \times 0 = 0$; $8 \times 0 = 0$; $12 \times 0 = 0$.

7. 8 exceeds 5 by $8-5$; 13 exceeds 7 by $13-7$; A exceeds B by ....

8. What must you add to 5 to make 8 ? Answer, $8-5$.
What must you add to 9 to make 15 ? Answer, $15-9$.
What must you add to 4 to make N ?
What must you add to A to make 12 ?
What must you add to P to make Q ?

9. You obtain 20 if you multiply 5 by $\frac{20}{5}$.
You obtain 36 if you multiply 2 by ....
You obtain 24 if you multiply N by ....
You obtain c if you multiply 10 by ....
What is the general statement ?

10. $\frac{4\times 7}{5\times 7}=\frac{4}{5}$; $\frac{3\times 11}{8\times 11}=\frac{3}{8}$; $\frac{6\times 9}{7\times 9}=\frac{6}{7}$.

11. $2\times 5=10$ and this is an even number.
$2\times 13=26$ and this is an even number.
If N is any whole number, $2\times N$ is ....

12. $2\times 4+1$ is odd; $2\times 11+1$ is odd.
If N is any whole number, ...

13. $2\times 6-1$, $2\times 9-1$, $2\times 20-1$ are all odd numbers.

14. Since $12+5=17$, $\therefore$ $12=17-5$.
Since $9+11=20$, $\therefore$ $9=20-11$.
If $a+b=c$, then ....

15. Since $4\times 5=20$ $\therefore$ $4=\frac{20}{5}$; since $9\times 6=54$, $\therefore$ $9=\frac{54}{6}$.
If $N\times 3=21$, then N .... If $a\times b=c$, then a ....

16. $5-1$, 5, $5+1$ are consecutive whole numbers.
$14-1$, 14, $14+1$ are consecutive whole numbers.

17. $5-2$, 5, $5+2$ are consecutive odd numbers.
$13-2$, 13, $13+2$ are consecutive odd numbers.
If N is ....

18. $6-2$, 6, $6+2$ are consecutive even numbers.
$14-2$, 14, $14+2$ are consecutive even numbers.

19. What is the next even number (i) above 18, (ii) above N if N is even, (iii) above N if N is odd ?

20. What is the next odd number, (i) above 22, (ii) above N if N is even, (iii) above N if N is odd ?

21. What is the odd number (i) just below 21, (ii) just below t, if t is even ?

22. The greatest of four consecutive odd numbers is N. What is the least ?

23. How many whole numbers are there *between* (i) 6 and 10, (ii) 3 and 11, (iii) the whole numbers x and y, if $y>x$?

24. Write down 5 consecutive whole numbers, such that the middle number is (i) 6, (ii) 10, (iii) the whole number p.

25. Up to and including 8, there are $\frac{8}{2}$ even numbers.
Up to and including 20, there are $\frac{20}{2}$ even numbers.
What is the general statement ?

26. Up to and including 9, there are $\frac{10}{2}$ odd numbers.
Up to and including 15, there are $\frac{16}{2}$ odd numbers.
What is the general statement ?

27. The sum of 4, 5, 6 is three times 5.
The sum of 11, 12, 13 is three times 12.
The sum of N, N + 1, N + 2 is ....
The sum of $a-1$, a, $a+1$ is ....

28. The sum of 5 and 7 is twice 6.
The sum of 16 and 18 is twice 17.
What is the general statement ?

29. Can you say at a glance the number of crosses in each of the groups (i), (ii), (iii) in Fig. 36 ? How many are there in the various compartments of each group ?

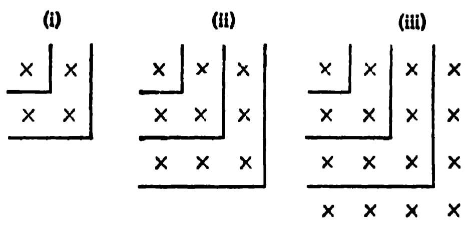


FIG. 36.

What is the value of (i) $1+3$, (ii) $1+3+5$, (iii) $1+3+5+7$?
Draw a group containing 5 rows of crosses with 5 crosses. in each row and divide it up in the same way. What is the value of $1+3+5+7+9$?
What is the sum of (i) the first 6 odd numbers, (ii) the first 20 odd numbers, (iii) the first n odd numbers ?

30. Take a number of groups of crosses as in Fig. 36 and add them up *along diagonals*. Prove in this way from group (iii) that $1+2+3+4+3+2+1=4^2$. Can you generalise this result ?

[*Note. For additional drill-examples, see Exercise E.P.* 2, p. 134. *For a revision exercise on Ch. I-II, see Appendix, Ex. R.* 1, *p.* 257.]

CHAPTER III

EASY PROBLEMS AND EQUATIONS

General Instructions

(i) Numbers may be represented by letters. **Do not use letters to represent quantities,** *i.e.* numbers-of-things. (See p. 10.)

(ii) In any problem which deals with quantities, **state your units clearly.**

(iii) Whenever you use a letter to represent an unknown number, first **write down a sentence, stating exactly what this letter represents.**

Example 1. Express by an equation the following statement: I think of a number; I then double it and add 7 to it; the result is 25.

We might say

$$(\text{Twice the number thought of}) + 7 = 25.$$

But it is simpler to say: denote the number thought of by n.

Then $$2n + 7 = 25.$$

The form in which this statement is now written is called an **equation,** and n is called "the unknown." The process of discovering the unknown number is called **solving the equation**; and the value of the unknown number is called the **root of the equation.**

The reader may be able to see what this value is; methods for finding it will be given later.

Example 2. The weight of a box and its contents is 10 lb.; the contents weigh 3 lb. more than the box.

Express these facts by an equation.

Suppose that the box by itself weighs W lb.

Then the contents of the box weigh (W + 3) lb.

$$\therefore \text{ W lb.} + (\text{W} + 3)\text{ lb.} = 10\text{ lb.}$$

This is a relation between quantities; we can obtain from it a relation between numbers, $W + W + 3 = 10$.

$$\therefore 2W + 3 = 10.$$

EXERCISE III. a

Criticise and correct, *where necessary*, the following statements :

1. Let the weight of the man be w.
2. Suppose the length of the room is l.
3. Let x be the cost of the house.
4. The distance of A from B is s miles.
5. Let the speed of the train be v.
6. Eggs are sold at N for a shilling.
7. Let $\angle ABC = x$.
8. Let t be the time it takes to walk a mile.
9. Let the required even number be N.
10. If the radius of a circle is r, its diameter is $2r$.
11. $x + y = 180°$.
12. If a room is l ft. long, b ft. broad, h ft. high, its volume is $l \times b \times h$.
13. His age is x.
14. Let the price of butter be y.
15. If n is an odd number, $n + 3$ is an even number.

EXERCISE III. b

What expressions represent the results in Nos. 1-16 ?

Think of a number n,

1. Double it.
2. Add 2 to it.
3. Halve it.
4. Subtract 2 from it.
5. Divide it by 10.
6. Diminish it by 4.
7. Multiply it by 3.
8. Increase it by 3.
9. Subtract it from 12.
10. Square it.
11. Take two-thirds of it.
12. Divide 24 by it.
13. Multiply it by 3 and subtract 3 from the result.
14. Add to it one half of itself.
15. Increase it by 5 and halve the result.
16. Diminish it by 3 and multiply the result by 4.

Write the statements in Nos. 17-25 as equations. If you can say *at sight* what the root of the equation is, do so.

17. I think of a number, then subtract 9 ; the result is 13.
18. I think of a number, then double it ; the result is 38.
19. I think of a number, divide it by 4, add 7 ; the result is 12.

20. I think of a number, then add 15; the result is the same as multiplying the original number by 4.

21. From the cube of a number, I subtract the square of the same number; the result is 180.

22. The sum of two consecutive numbers is 37.

23. The difference between a number and its square is 72.

24. I think of a number, add 5, double the result, then subtract the original number. This gives 17.

25. A number exceeds 5 by half the amount that it falls short of 17.

Write the statements in Nos. 26-37 as equations. *State clearly in each case what the unknown represents.* If you can say at sight or with very little work what the root of the equation is, do so.

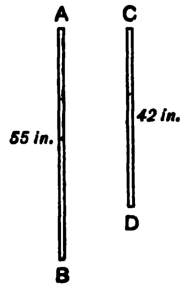


FIG. 37.

26. A scuttle of coal weighs 28 lb.; the coal weighs three times as much as the scuttle.

27. The rods AB and CD in Fig. 37 are cut down by equal amounts, so that one is just twice the length of the other.

28. A is now 53 years old and B is 21; in a certain number of years' time, A will be just twice as old as B.

29. In Fig. 38, BP exceeds PC by $1\frac{1}{2}$ inches and BC is 5 inches.

30. In Fig. 38,

$$\angle BAC = 4\angle ABC = 4\angle ACB.$$

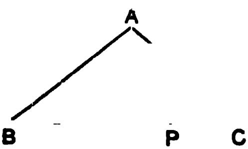


FIG. 38.

31. In Fig. 38, $AB = AC = \frac{2}{3}BC$, and the perimeter of the triangle ABC is 10·5 inches.

32. A hall is twice as broad and three times as long as it is high; it contains 6000 cu. ft. of air.

33. One tap pours water into a bath twice as fast as another tap; it takes 108 gallons to fill the bath; this is done, by both taps together, in 6 minutes.

34. The sum of the angles of a certain polygon is 14 right-angles. [If a polygon has n sides, the sum of its angles is $(2n-4)$ right-angles.]

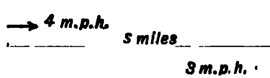


FIG. 39.

35. A man walks at 4 miles an hour from his house to a station, see Fig. 39, and returns home at 3 miles an hour; the two journeys together take $3\frac{1}{2}$ hours.

36. The boundary of the shaded area in Fig. 40, formed by cutting a quadrant away from a square is 5 inches. [Take the length of the circumference of a circle of radius r inches as $\frac{44r}{7}$ inches.]

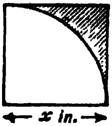


FIG. 40.

37. The shaded area in Fig. 40 (see No. 36) is $10\frac{1}{2}$ sq. inches. [Take the area of a circle of radius r inches as $\frac{22r^2}{7}$ sq. inches.]

Give statements about numbers which correspond to the equations in Nos. 38-45.

38. $n+6=33$. **39.** $n-7=19$.

40. $n+5n=42$. **41.** $n+\frac{n}{4}=20$.

42. $20-n=9$. **43.** $n+18=3n$.

44. $n+(n+1)+(n+2)=27$. **45.** $2n^3-3n=1000$.

Solving Equations

Example 3. What is n, if $n+6=19$?

I think of a number; then add 6; the result is 19.

By adding 6, I obtain 19; $\therefore$ the number $=19-6=13$.

$$\therefore\ n=13.$$

Check: $n+6=13+6=19$.

Example 4. What is x, if $7x=63$?

I think of a number; then multiply it by 7; the result is 63.

By multiplying by 7, I obtain 63; $\therefore$ the number $=\frac{63}{7}=9$.

$$\therefore\ x=9.$$

Check: $7x=7\times 9=63$.

EXERCISE III. c

Express each equation as a "think of a number" problem; then find the answer and check it.

1. $n+8=17$. **2.** $n-5=11$ **3.** $3y=21$.

4. $\frac{z}{2}=9$. **5.** $x+1\frac{1}{2}=2$. **6.** $n-2\frac{1}{2}=5$.

7. $N-7=0$. **8.** $8x=0$. **9.** $\frac{y}{5}=7$.

10. $z+2z=18$. **11.** $3n-2=10$. **12.** $\frac{x}{3}+4=9$.

13. $4y-y=21$. **14.** $\frac{z+2}{4}=5$. **15.** $\frac{z}{4}+2=5$.

General Methods for solving Simple Equations

Example 5. Solve $n-17=46$.

Since the numbers $n-17$ and 46 are equal, if we *add* 17 *to each of them*, the results will be equal.

$$\therefore\ n-17+17=46+17;$$
$$\therefore\ n=63.$$

Example 6. Solve $x+17=46$.

Since the numbers $x+17$ and 46 are equal, if we *subtract* 17 *from each of them*, the results will be equal.

$$\therefore\ x+17-17=46-17;$$
$$\therefore\ x=29.$$

Example 7. Solve $7y=91$.

Since the numbers $7y$ and 91 are equal, if we *divide each number* by 7, the results will be equal.

$$\therefore\ 7y\div 7=91\div 7.$$
$$\therefore\ y=13.$$

Example 8. Solve $\frac{z}{3}=14$.

Since the numbers $\frac{z}{3}$ and 14 are equal, if we *multiply each number by* 3, the results will be equal.

$$\therefore\ \frac{z}{3}\times 3=14\times 3;$$
$$\therefore\ z=42.$$

The solution of every simple equation is performed by applying one or more of these four arguments.

Example 9. Solve $3x-12=\frac{2x}{3}+16$.

Add 12 to each side, $\therefore\ 3x-12+12=\frac{2x}{3}+16+12;$

$$\therefore\ 3x=\frac{2x}{3}+28.$$

Subtract $\frac{2x}{3}$ from each side, $\therefore\ 3x-\frac{2x}{3}=\frac{2x}{3}+28-\frac{2x}{3};$

$$\therefore\ 3x-\frac{2x}{3}=28.$$

Multiply each side by 3, $\therefore\ 3x \times 3 - \frac{2x}{3} \times 3 = 28 \times 3$;

$$\therefore\ 9x - 2x = 84;$$

$$\therefore\ 7x = 84.$$

Divide each side by 7, $\therefore\ x = 12$.

Check: Left side $= 3x - 12 = 3 \times 12 - 12 = 36 - 12 = 24$.

Right side $= \frac{2x}{3} + 16 = \frac{2 \times 12}{3} + 16 = 8 + 16 = 24$.

$\therefore$ when $x = 12$, left side = right side.

The four arguments used in solving simple equations may be summarised as follows:

(i) *Equal numbers may be added to each side.*
(ii) *Equal numbers may be subtracted from each side.*
(iii) *Each side may be multiplied by equal numbers.*
(iv) *Each side may be divided by equal numbers.*

From (i), if $x - a = b$, then $x = b + a$.

From (ii), if $x + a = b$, then $x = b - a$.

This shows that **any term may be moved from either side of an equation to the other side if the sign in front of it is changed.**

It is sometimes convenient to reverse the order of an equation. Thus, if $3 = x$, we can at once say $x = 3$, without using any of the arguments (i)-(iv) above.

Example 10. Solve $\frac{5}{x} = \frac{2}{3}$.

Multiply each side by $3x$, $\therefore\ \frac{5}{x} \times 3x = \frac{2}{3} \times 3x$;

$$\therefore\ 15 = 2x;\ \therefore\ 2x = 15.$$

Divide each side by 2, $\therefore\ x = \frac{15}{2} = 7\frac{1}{2}$.

When solving an equation

Start by writing down the equation *exactly as it stands in the book*: do not try to simplify it in your head.

Be careful not to confuse the symbols $=$ and $\therefore$.

$=$ means "is equal to"; $\therefore$ means "therefore."

Thus we say: $3x = 15$; $\therefore\ x = 5$.

When checking an equation

Substitute for " the unknown " in each side *separately*, as in Example 9.

Substitute in the equation as it is given, *not* in any simplified form of it. The object of checking is to make sure that your answer is right. You cannot be certain that it is, unless you substitute in the actual equation given you.

EXERCISE III. d

Solve the equations in Nos. 1-10, explaining each step in the argument as in Examples 9, 10 above : and check each answer.

1. (i) $3n = 21$; (ii) $8x = 64$;
(iii) $1\frac{1}{2}R = 12$; (iv) $0{\cdot}3m = 1{\cdot}2$.

2. (i) $a - 4 = 7$; (ii) $p - 7 = 0$;
(iii) $z - 2\frac{1}{2} = 6\frac{1}{2}$; (iv) $w - 3{\cdot}2 = 1{\cdot}9$.

3. (i) $l + 5 = 12$; (ii) $x + 2\frac{1}{2} = 7$;
(iii) $t + \frac{2}{5} = 1$; (iv) $t + 7{\cdot}4 = 9$.

4. (i) $\frac{1}{8}p = 3$; (ii) $\frac{y}{4} = 7$;
(iii) $\frac{n}{3} = \frac{2}{7}$; (iv) $\frac{q}{5} = 4{\cdot}2$.

5. (i) $\frac{4R}{5} = 2$; (ii) $\frac{3p}{8} = 12$;
(iii) $\frac{2m}{3} = 10$; (iv) $\frac{3x}{5} = 1$.

6. (i) $3 = n - 2$; (ii) $7 = p + 5$;
(iii) $4\frac{1}{2} = t + 1\frac{1}{4}$; (iv) $6{\cdot}2 = 3{\cdot}8 + k$.

7. (i) $3y - 4 = 8$; (ii) $4R + 3 = 13$;
(iii) $5N + 2 = 17$; (iv) $7z - 3 = 25$.

8. (i) $6y - 15 = 0$; (ii) $5y + 2 = 7\frac{1}{2}$;
(iii) $7l - 3 = 9$; (iv) $9t + 8 = 20$.

9. (i) $0{\cdot}3x = 6$; (ii) $2{\cdot}5t = 11$;
(iii) $1{\cdot}6z = 12$; (iv) $0{\cdot}5k = 0$.

10. (i) $\frac{3y}{2} = 1$; (ii) $\frac{4a}{7} = \frac{2}{5}$;
(iii) $\frac{3t}{8} = 0$; (iv) $\frac{5t}{8} = \frac{7}{5}$.

Solve the following equations and check each answer.

11. $2p-8=p-3$. **12.** $2l+4=19-l$. **13.** $t+7=17-4t$.

14. $3(n-7)=12$. **15.** $4(2k+1)=20$. **16.** $7(3y-1)=28$.

17. $4(t-5)=0$. **18.** $l-\frac{1}{3}l=6$. **19.** $m+\frac{m}{5}=24$.

20. $5=3\text{R}$. **21.** $0=2t-7$. **22.** $10y=y$.

23. $x-\frac{2x}{7}=10$. **24.** $\frac{p}{2}-\frac{p}{3}=1$. **25.** $\frac{l}{3}=1+\frac{l}{4}$.

26. $\frac{\text{R}}{5}-\frac{2}{7}=0$. **27.** $\frac{1}{2}(3x-1)=7$. **28.** $3=\frac{1}{4}(2x+1)$.

29. $3t=5{\cdot}7$. **30.** $\frac{1}{4}k=1{\cdot}7$. **31.** $2{\cdot}4y=6$.

32. $\frac{3}{x}=\frac{2}{5}$. **33.** $\frac{3}{4}=\frac{2}{z}$. **34.** $\frac{5}{2p}=\frac{1}{6}$.

35. $\frac{2\text{R}}{3}-\frac{\text{R}}{2}=1$. **36.** $\frac{t}{3}+\frac{t}{5}=0$. **37.** $\frac{2y-3}{5}=3$.

38. $\frac{6x}{7}+2=11$. **39.** $5=\frac{3k-1}{4}$. **40.** $2{\cdot}7=\frac{3}{4}m$.

41. $\frac{t+1}{5}=\frac{t+3}{6}$. **42.** $\frac{y-1}{3}=\frac{2y+1}{7}$. **43.** $\frac{2\text{R}+4}{5}=\frac{4-\text{R}}{3}$.

44. $\frac{3x}{5}-\frac{x}{2}=\frac{1}{2}$. **45.** $1+\frac{7a}{2}=a+6$. **46.** $\frac{1}{z}+\frac{1}{3z}=\frac{1}{12}$.

47. $\text{R}-0{\cdot}7\text{R}=12$. **48.** $\frac{x}{0{\cdot}3}=4$. **49.** $\frac{y}{3}\div 1\frac{1}{2}=1\frac{1}{3}$.

[*Note. For additional drill-examples, see Exercise E.P.* 3, *p.* 135.]

General Procedure in the Solution of Problems

1. *Read the question carefully.* Do not start to try to work it out before you are sure that you understand what you are given and what you are asked to find out.

2. Take a letter to stand for some unknown *number* which the problem involves.

If the problem involves quantities, ***state clearly what the unit is.***

Never say, let the length be x ; a clear statement would be, let the length of the room be x feet.

Never say, let x be the cost of eggs ; a clear statement would be, suppose one dozen eggs cost x pence.

3. *Check the answer by using the actual data of the problem.*

It is not sufficient to check by substituting in the equation, because your equation may be wrong.

Example 11. Share 10 shillings between two boys, A and B, so that A receives 1s. 6d. more than B.

Suppose that B's share is b shillings.

Now A receives $1\frac{1}{2}$ shillings more than B.

$$\therefore \text{ A's share is } (b + 1\tfrac{1}{2}) \text{ shillings.}$$

But the total sum shared out is 10 shillings ;

$$\therefore (b + 1\tfrac{1}{2}) \text{ shillings} + b \text{ shillings} = 10 \text{ shillings ;}$$

$$\therefore (b + 1\tfrac{1}{2}) + b = 10 ;$$

$$\therefore 2b + 1\tfrac{1}{2} = 10 ; \quad \therefore 2b = 10 - 1\tfrac{1}{2} = 8\tfrac{1}{2} ;$$

$$\therefore b = 4\tfrac{1}{4}.$$

$$\therefore \text{ B's share} = 4\tfrac{1}{4}\text{s.} = 4\text{s. } 3\text{d.}$$

$$\therefore \text{ A's share} = 4\text{s. } 3\text{d.} + 1\text{s. } 6\text{d.} = 5\text{s. } 9\text{d.}$$

Check : The total sum shared out equals 4s. 3d. + 5s. 9d. = 10s.

EXERCISE III. e

Solve the following problems by Algebra, and check each answer.

1. I think of a number, divide it by 4 and add 11 ; the result is 17. What is the number ?

2. I think of a number, add to it one-third of itself ; the result is 28. What is the number ?

3. The result of adding 42 to a certain number is the same as multiplying that number by 4. What is the number ?

4. If I halve a certain number and add 1, the result is the same as dividing the number by 3 and adding 4. What is the number ?

5. The sum of two consecutive numbers is 55. What are they ?

6. What number exceeds 17 by the same amount as it falls short of 55 ?

7. The sum of three consecutive even numbers is 72 ; what are they ?

8. From three-quarters of a certain number, 3 is subtracted ; the result is two-thirds of that number. What is the number ?

9. I think of a number, add 2 to it, multiply the sum by 5 and then subtract 7 ; the result is 23. What is the number ?

10. I think of a number, double it, add 12, then divide by 2. I now subtract the number I first thought of; the result is 6. Can you find the original number ?

11. In Fig. 41, find AP if AP is 3 inches longer than PB.

FIG. 41.

12. In Fig. 41, find AP if AP is twice the length of PB.

13. In Fig. 41, find AQ if AQ is $2\frac{1}{2}$ times as long as BQ.

14. In Fig. 42, find $\angle$AOC if $\angle$BOC $= 5\angle$AOC.

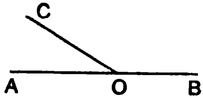


FIG. 42.

15. In Fig. 42, find $\angle$AOC if $\angle$BOC exceeds twice $\angle$AOC by 90°.

16. With the data of Fig. 43, find x.

17. An excursion ticket is one-quarter of the ordinary fare. I save 5s. 6d. by taking an excursion ticket. What is the ordinary fare ?

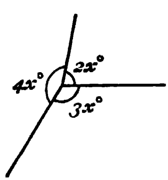


FIG. 43.

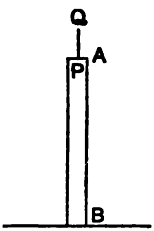


FIG. 44.

18. The flagstaff PQ in Fig. 44 is one-fifth of the height AB of a tower ; Q is 90 feet above the ground. What is AB ?

19. In Fig. 45, find $\angle$B if $\angle B = \angle C = 4\angle A$.

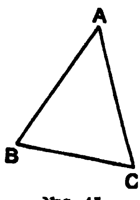


FIG. 45.

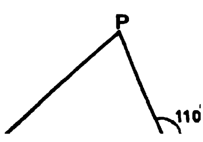


FIG. 46.

20. In Fig. 46, find $\angle$P if $\angle P = 3\angle Q$.

21. The sum of the angles of an n-sided polygon is $(2n-4)$ right angles. How many sides has a polygon if its angle-sum is 20 right angles ?

22. I buy a house, and have to spend one-third as much on repairs. The total cost is £3000. What did the house cost ?

23. Fig. 47 represents a hurdle whose width is $1\frac{1}{2}$ times its height. It is made of metal strips, whose total length is 36 feet. How high is the hurdle ?

Fig. 47.

24. Fig. 48 represents a skeleton wire cage ; AB is twice as long as BC and as CD. The total length of wire is $5\frac{1}{4}$ ft. What are the lengths of AB and BC ?

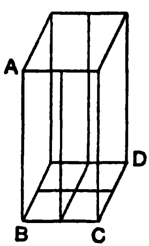


Fig. 48.

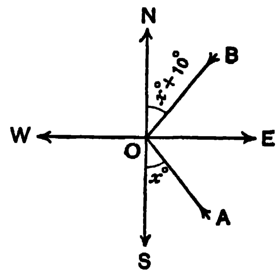


Fig. 49.

25. The wind backs from direction AO to direction BO, see Fig. 49 ; if the change of direction is 100°, find its first direction.

[*Note. For additional examples, see Appendix, Ex S.* 4, *p.* 283.

TEST PAPERS. A, 1-8

A. 1

1. Write more shortly :

(i) $2a \times 4$; (ii) $3b \times 3bc$; (iii) $\frac{3t}{2} \times 4$.

2. If $y = 3x - 1$, find (i) the value of y if $x = 4$;
(ii) the value of x if $y = 14$.

3. Solve the equations, (i) $t + \frac{3}{4}t = 3\frac{1}{2}$;
(ii) $7 = \frac{1}{3}(2W + 3)$.

4. A boy is now k years old and his father is $4k$ years old. How old will the father be when the boy is $2k$ years old ?
What is k if the boy was 1 year old when his father was 25 ?

5. Fig. 50 represents a rectangle. Find the *numerical* value of (i) its perimeter, (ii) its area.

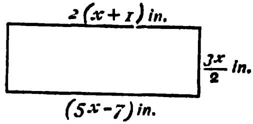


FIG. 50.

A. 2

1. Simplify :

$$\text{(i) } n-\tfrac{3}{5}n\ ;\ \text{(ii) } 5y+y-2y.$$

2. (i) If $y=2x^2-3x$ and if $x=4$, find y.

(ii) Subtract $9r$ inches from $2r$ feet. Answer in yards.

3. (i) A clock loses n minutes per week ; express this loss in seconds per hour.

(ii) If a clock loses t seconds an hour, it loses T minutes a week, express t in terms of T.

4. Solve the equations :

(i) $\dfrac{3\mathrm{N}}{8}=9$;

(ii) $\dfrac{2r}{5}-\dfrac{r}{3}=1$;

(iii) $p+4=q-7=20$.

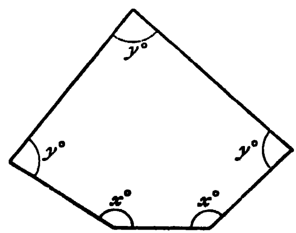


FIG. 51.

5. With the data of Fig. 51, find x if $x=1\frac{1}{2}y$.

A. 3

1. Write more shortly :

$$\text{(i) } 3b\times 3ab\ ;\ \text{(ii) } \frac{c}{2}\times 6c\ ;\ \text{(iii) } 4p\div\tfrac{1}{2}.$$

2. A tram fare is 3d. ; what is the cost of n journeys, (i) in pence, (ii) in shillings ?

3. (i) Simplify $2a(3a-5b)+5b(2a-3b)$.

(ii) What is the value of x if $xy+2y^2=xy^2+2y$ when $y=3$?

4. Fig. 52 represents two people leaving A and B at 10 a.m.

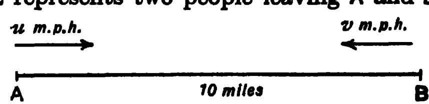


FIG. 52.

How far apart are they at (i) 11 a.m., (ii) 10.30 a.m. ?

What can you say about u, v if they meet at 11.15 a.m. ?

5. Find two consecutive numbers such that one-quarter of the smaller exceeds one-fifth of the larger by 4.

A. 4

1. If $p=1$, $q=2$, $r=3$, $s=4$, find the values of:

(i) $p+q(r+s)$; (ii) $(p+q)r+s$.

2. The square of $x+\frac{1}{x}$ is $x^2+\frac{1}{x^2}+2$. Show that this is true (i) when $x=1$, (ii) when $x=3$.

3. Solve the equations:

(i) $\frac{n}{4}=\frac{3}{5}$; (ii) $4-z=1{\cdot}8$; (iii) $17-x=2(2+x)$.

4. The circumference of the wheel of a car is p feet; how many revolutions per minute does the wheel make when the car is travelling v yards per second?

5. A has £100, B has £50: but after B has paid A what he owes him, A has three times as much as B. What was the debt?

A. 5

1. (i) Simplify $7c-2c+3c$.

(ii) If $p=2q$ and $q=5s$, find the value of pq when $s=\frac{1}{2}$.

2. Express in florins the difference between £x and $3x$ half-crowns.

3. Solve the equations:

(i) $\frac{3}{n}=\frac{2}{7}$; (ii) $\frac{3}{4}(y+5)=9$.

If $4(x+a)=5(x-a)+4$, find a when $x=2$.

4. A hotel bill for n days is $15n$ shillings if $n<4$, and is $12(n+1)$ shillings if $n>4$. What is the bill for 3 days and for 5 days? What would you expect the bill to be for 4 days?

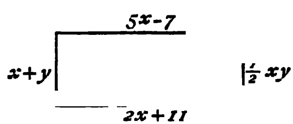


FIG. 53.

5. Fig. 53 shows the lengths of the sides of a rectangle in inches.

What are their numerical values?

A. 6

1. If $p=1$, $q=\frac{1}{2}$, find the values of

(i) $2pq$; (ii) p^2+q^2; (iii) $\frac{p+q}{p-q}$.

2. I buy 6N eggs; how many are left after 6 have been eaten? What is N if one dozen are left?

3. Solve the equations, (i) $l=3(10-l)$;
(ii) $2{\cdot}7x-3{\cdot}4=1{\cdot}9x+1{\cdot}4$.

4. Find in terms of x the third angle of the triangle in Fig. 54. Show that the triangle is isosceles if $x=10$ or 34 or 40.

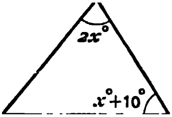


FIG. 54.

5. Find t if $t°$ Centigrade is the same temperature as $2t°$ Fahrenheit. (Use the fact stated in Ex. II. c, No. 14, p. 23.)

A. 7

1. If $b=3$, $c=0$, $d=5$, find the values of

$$\text{(i) } \frac{d+1}{b};\ \text{(ii) } bc-c;\ \text{(iii) } (b+1)(d-1).$$

2. Simplify (i) $t+3t+t$; (ii) $3k\times 2k$; (iii) $9p\div 15$.

3. Solve the equations:

$$\text{(i) } \frac{5}{3}=\frac{2}{w};\quad \text{(ii) } \frac{x-1}{2}=\frac{3x+1}{11}.$$

Prove that $R^2+8r^2=6Rr$ if $R=6$ and $r=3$.

4. On a holiday, a man walks a certain distance the first day, and half as far again the second day. He walked s miles the second day; how far did he walk the first day?

5. A pyramid stands on an n-sided base. How many more edges than corners has it got?

What is n if the sum of the number of edges and number of corners is 25?

A. 8

1. (i) Add 3N to 5N and divide the result by 6.
(ii) Multiply $2p$ by $6p$.

2. (i) If $W=5w$, simplify $\dfrac{W-w}{W+w}$.

(ii) How many $2\frac{1}{2}$d. stamps can be bought for P half-crowns?

3. Solve the equations:

$$\text{(i) } \frac{p}{5}=2{\cdot}4;\ \text{(ii) } t-0{\cdot}3t=3{\cdot}5;\ \text{(iii) } \tfrac{1}{2}(3z-7)=4.$$

4. A closed tin box is $3c$ in. long, $2c$ in. broad, c in. high. What is the sum of the lengths of its edges (i) in inches (ii) in feet? What is the total surface area in sq. inches?

5. O is the centre of the circle in Fig. 55; the arcs AQB and APB are of lengths $(x-2)$ inches and $(4x+2)$ inches. Find the numerical value of the length of the circumference of the circle.

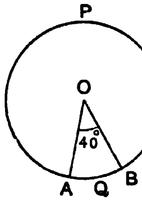


FIG. 55.

CHAPTER IV

ELEMENTARY PROCESSES

Like Terms

AN expression of the form $5a - 3a + 4a$ is said to consist of *like terms*, because it can be reduced to a single term.

Just as 5 dozen − 3 dozen = (5 − 3) dozen = 2 dozen,

and 5 dozen − 3 dozen + 4 dozen = 2 dozen + 4 dozen = 6 dozen,

so
$$5a - 3a = (5 - 3)a = 2a,$$
and
$$5a - 3a + 4a = 2a + 4a = (2 + 4)a = 6a.$$

When simplifying, *work from the left* unless brackets or $\times$ and $\div$ signs show that operations must be performed in a different order.

Example 1. Simplify $4a - \frac{1}{2}a$.

$$4a - \tfrac{1}{2}a = (4 - \tfrac{1}{2})a = 3\tfrac{1}{2}a,$$

or
$$4a - \tfrac{1}{2}a = \frac{4a}{1} - \frac{a}{2} = \frac{8a - a}{2} = \frac{7a}{2}.$$

These two results are equivalent because

$$3\tfrac{1}{2}a = 3\tfrac{1}{2} \times a = \frac{7}{2} \times a = \frac{7a}{2}.$$

Example 2. Simplify $b + \frac{2b}{3} - \frac{b}{6}$.

$$b + \frac{2b}{3} - \frac{b}{6} = \frac{6b + 4b - b}{6} = \frac{9b}{6} = \frac{3b}{2}.$$

EXERCISE IV. a

Simplify the following expressions:

1. $3a + 7a - 5a$.
2. $2b + 9b - b - 3b$.
3. $3c + 7c - 5c + c$.
4. $6d - 3d - 2d$.
5. $2e - \frac{1}{2}e$.
6. $f - \frac{f}{4}$.
7. $1\frac{1}{10}g + 2\frac{1}{2}g$.
8. $\frac{h}{2} + \frac{h}{3}$.
9. $\frac{k}{3} + \frac{k}{6}$.
10. $\frac{2l}{3} - \frac{l}{6}$.
11. $2m + m \times 3$.
12. $2n \times 5 - n$.

13. $9p - 3 \times 2p$. 14. $3 \times 3q + q$. 15. $2r \times 3 - 3r \times 2$.

16. $6st - 3st - 2st$. 17. $5v^2 - 2v^2 + 3v^2$. 18. $3xy - xy$.

19. $4yz + 4zy$. 20. $6z^2 - 2z^2 - 4z^2$. 21. $\frac{2a}{5} + a + \frac{a}{10}$.

22. $\frac{9b}{12} - \frac{b}{4}$. 23. $\frac{2c}{5} + 1\frac{1}{2}c - \frac{1}{10}c$. 24. $3\frac{1}{2}d - 1\frac{3}{4}d + \frac{d}{4}$.

25. $10ef - 7fe - \frac{2}{3}ef$. 26. $\frac{3}{4}g^2 + g^2 - \frac{2g^2}{3}$.

27. $h + h + h + h + h + h + h + h$.

28. $l + l + l + l + \ldots$ fifteen terms.

29. $3m + 3m + 3m + \ldots$ ten terms.

30. $2p^2 + 2p^2 + 2p^2 + \ldots$ eight terms.

[*Note. For additional drill-examples, see Exercise E.P. 4, p.* 136.]

Unlike Terms

Example 3. A tourist walked at v miles an hour for 4 hours on the first day and at v miles an hour for t hours on the second day. How far did he walk in the two days ?

On the first day, he walked $4v$ miles.

On the second day, he walked tv miles.

$\therefore$ the total distance is $(\mathbf{4v + tv})$ **miles.**

In this answer, $4v$ and tv are *unlike terms* ; we cannot simplify the expression unless the numerical value of t is given.

We can of course say that the tourist walked in the two days for $(4 + t)$ hours at v miles an hour, and therefore the total distance is $(4 + t)v$ miles or $v(4 + t)$ miles. Here, we say that 4 and t are unlike terms, because $4 + t$ cannot be written more simply unless the value of t is known.

The expressions $4v + tv$, $(4 + t)v$ and $v(4 + t)$ are equal, and each involves the sum of two unlike terms.

Example 4. Find the short-hand form of

$$3a + 2 + a + 8b - 2a - 3b.$$

In this expression, $3a + a - 2a$ are like terms, also $8b - 3b$ are like terms.

$$\therefore \text{ the expression} = 3a + a - 2a + 8b - 3b + 2$$
$$= \mathbf{2a + 5b + 2.}$$

There is no shorter way of writing this expression, if a and b represent any numbers whatever.

EXERCISE IV. b

1. What is the length of a fence formed by 10 hurdles each r feet long and 12 hurdles each s feet long ?

2. What is the total bill for c lb. of tea at 2s. per lb. and p lb. of sugar at 4d. por lb. ? Answer (i) in pence, (ii) in shillings.

3. Each mesh in Fig. 56 is l in. long and b in. broad. What is the total length of wire required for the network ?

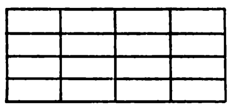


FIG. 56.

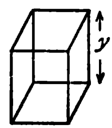


FIG. 57.

4. Fig. 57 represents a skeleton box with base x inches square and y inches high. What length of wire is used in making it ?

5. A boy works for p hours on Sunday, q hours on each Wednesday and on each Saturday, and r hours on each other day. How many hours does he work each week ?

6. From a rod $(3l+2m)$ inches long, a portion $2l$ inches long is cut off ; what is the length of the remainder ?

7. Subtract $7x$ pence from x shillings. Answer in pence.

8. Subtract k florins from £g. Answer in shillings.

9. Fig. 58 shows the lengths of the sides of a triangle in inches. (i) What is the perimeter ? (ii) By how much does AB + AC exceed BC, and BA + BC exceed AC ?

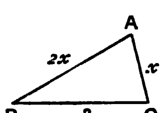


FIG. 58.

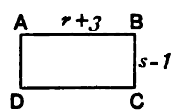


FIG. 59.

10. Fig. 59 shows the lengths of two adjacent sides of a rectangle in inches. (i) What is the perimeter ? (ii) What is the length of AD + DC + CB ?

11. If, in Fig. 59, $r=3s$, find the perimeter of the rectangle in terms of s (the units are inches).

12. A man starts with £$(x+y)$. He pays x bills of 5 shillings each and y bills of 15 shillings each. How many shillings has he left ?

13. A body A weighing $(6x+4)$ lb. is put in one scale pan of a weighing machine, and a body B weighing $2x$ lb. is put in the other scale pan. What weight must be added to make them balance ?

14. The outside measurements of a box without a lid are: length $3x$ in., breadth $2x$ in., height h in.; it is made of wood 1 inch thick. What are the inside measurements ?

15. Equal holes, each x in. long, y in. wide are punched in the metal sheet, shown in Fig. 60. What is the surface-area of the sheet ?

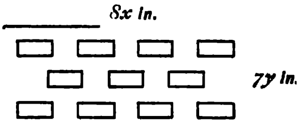


FIG. 60.

16. A rectangular garden is represented by the shaded area in Fig. 61; it is enlarged, as shown; the units of the given dimensions are yards. What is (i) the final area, (ii) the increase of area ?

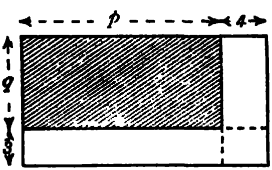


FIG. 61.

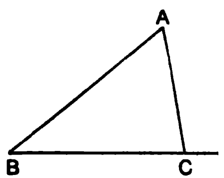


FIG. 62.

17. In Fig. 62, $\angle ABC = (x+2y)$ degrees, $\angle BAC = (y+2x)$ degrees. What is $\angle ACD$?

18. In Fig. 62, $\angle ACD = (3p+4q)$ degrees, $\angle ABC = (2p+q)$ degrees. What is $\angle BAC$?

Write down (when possible) short-hand forms for the expressions in Nos. 19-52. *If there is no shorter form, say so.*

19. $b+c+b$. **20.** $r+s+s+r$. **21.** $a+2+a$.

22. $l+m+l+m+l$. **23.** $x+2+y+3+x$. **24.** $e+e+e+3f$.

25. $4+4c$. **26.** $3t+2t+1$. **27.** $z+2+z+z+1$.

28. $R+r-R+r$. **29.** $10d-7d-2$. **30.** $3p-q+p$.

31. $1-t+t$. **32.** $2x+2y$. **33.** $6b-2b-3$.

34. $3a+4a+5a+6$. **35.** $6p+q-2p+q$. **36.** $3r-3s$.

37. $l+m-l$. **38.** $3e+2f-e-f$.

39. $2a\times 3-3b\times 2$. **40.** $3p+5q-q-2p$.

41. $r+s+t+t-s+r$.
42. $x+y+z+x-y-z$.
43. $3a+b+4c-a-b$.
44. $p+3q+3+3q-1$.
45. $3ab+3ac$.
46. $3rs+2st-sr$.
47. $3bc+3cb$.
48. $pq+p+q$.
49. $2xy+3x-yx$.
50. $5yz+2zy-yz-6yx$.
51. $ab+bc+abc$.
52. $x+2y+1-x+1$.
53. Add xy to x.
54. Subtract b from bc.
55. Subtract yz from zy.
56. Add xy to xz.
57. Increase p by 1.
58. Decrease $2q$ by 2.
59. Add $R+r$ to $3R$.
60. Add a to 0.
61. Simplify $11\times 23-10\times 23$.
62. Add 13×7 to 13×3.

[*Note. For additional drill-examples, see Exercise E.P.* 4, *p.* 136.]

Powers

The short-hand form of $x\times x\times x\times x$ is $\mathbf{x^4}$ and the short-hand form of $7\times x\times x\times x\times x$ is $\mathbf{7x^4}$.

The numerical factor 7 in the term $7x^4$ is called the **coefficient** of the term $7x^4$ or more shortly the **coefficient of $\mathbf{x^4}$**.

The term $7x^4$ is said to be of **degree 4 in x** or of the **4th degree in x**. The symbol x^4 is read as " x to the power 4 " and the 4 is called the **index** of x.

Thus, in the expression $2x^5+4x^3+x^2+3$, the term of degree 3 is $4x^3$ and its coefficient is 4, the coefficient of x^2 is 1, and the coefficient of x is 0, since this term is missing. The *numerical* term 3 is called the **constant term** or the **term independent of x**, because its value does not depend on the value of x. since it does not contain x.

Example 5. A shed A, $3x$ ft. long, $2x$ ft. wide, is built in a rectangular courtyard, leaving a passage 5 ft. wide along one side and 3 ft. wide along the other. What was the original area of the courtyard ?

The courtyard can be divided into four rectangles A, B, C, D. as shown in Fig. 63.

The area of A is $2x \times 3x$ sq. ft. $= 2 \times 3 \times x \times x$ sq. ft. $= 6x^2$ sq. ft.
The area of B is $5 \times 2x$ sq. ft. $= 10x$ sq. ft.
The area of C is $3 \times 3x$ sq. ft. $= 9x$ sq. ft.
The area of D is 3×5 sq. ft. $= 15$ sq. ft.

$\therefore$ the total area is $(6x^2 + 10x + 9x + 15)$ sq. ft.

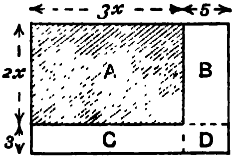


FIG. 63.

Now $10x$ and $9x$ are like terms ; $\therefore$ $10x + 9x = 19x$.

$\therefore$ the total area is $(\mathbf{6x^2 + 19x + 15})$ sq. ft.

Note. This expression cannot be written more shortly, unless we know the value of x, because $6x^2$ and $19x$ are *not* like terms.

$6x^2$ means $6 \times x \times x$ or $6x \times x$.

$19x$ means $19 \times x$.

Just as $7x + 3x = (7 + 3)x$, so

$$6x^2 + 19x = 6x \times x + 19 \times x = (6x + 19) \times x,$$

but this is no shorter, and still contains *two unlike terms*, viz. $6x$ and 19.

Expressions should always be written in an orderly way. They are usually arranged *either* in **descending powers** of the unknown, *i.e.* beginning with the highest power, then the next highest, and so on, *or* in **ascending powers**, *i.e.* beginning with the constant term, then the term of first degree, then the term of second degree, and so on.

Thus $2x^5 + 4x^3 + x^2 + 3$ is arranged in *descending* powers of x.
And $3 + x^2 + 4x^3 + 2x^5$ is arranged in ascending powers of x.

Example 6. Simplify $2y^2 + 3y + 2 - y^2 + 2y + 1$ and arrange it in (i) descending powers of y, (ii) ascending powers of y.

Take the like terms together.

$$2y^2 - y^2 = y^2 \ ; \ 3y + 2y = 5y \ ; \ 2 + 1 = 3 \ ;$$

$\therefore$ the expression $= \mathbf{y^2 + 5y + 3}$, descending powers
$= \mathbf{3 + 5y + y^2}$, ascending powers.

EXERCISE IV. c

Find the area of the figures in Nos. 1-9, all the corners being right-angled, the units of the given dimensions being inches.

Express the answers without brackets.

1.

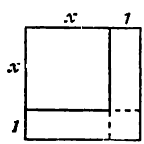


FIG. 64.

2.

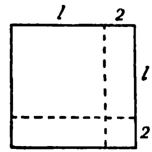


FIG. 65.

3.

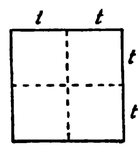


FIG. 66.

4.

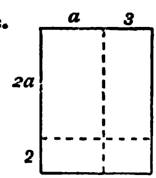


FIG. 67.

5.

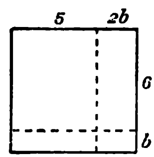


FIG. 68.

6.

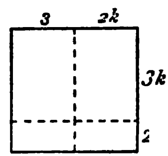


FIG. 69.

7.

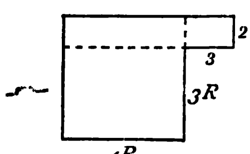


FIG. 70.

8.

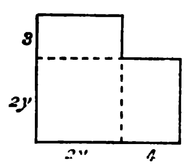


FIG. 71.

9.

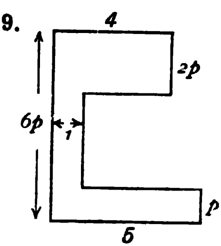


FIG. 72.

Simplify, where possible, the following expressions, and arrange them in descending powers :

10. $8a^2 - 4a^2$.

11. $3b + b^2 + b$.

12. $2c^2 + 6c - 3c - 3$.

13. $p^3 + 2p^3 - 3p$.

14. $t^2 - 1 + t - 1$.

15. $1 + r + 2r^4 - r$.

16. $5s^2 + 8s - 4s^2 - 8$.

17. $x^2 + 2x + 6 + 3x$.

18. $y^3 + y^2 + y + 1 + y + y^2$.

19. $5 + 6z + 3z^2 - 5z$.

20. $3 + 5k^2 - 2k - 1 + k^2$.

21. $3e^2 + e^4$.

22. $7b^3 - 3b - b^3$.

23. $3 + t^3 + 3t^2$.

24. $3x^2 - 5x + 7 - x^2 + 1$.

25. $6 + 6y^3 - 5y^2 - 5y$.

26. $3a^3 - 3$.

27. $b^3 + 4b^2 - b^3 - 4$.

28. $2c^3 - \frac{1}{2}c + 3c^3 + c$.

29. $5 + \frac{3t}{2} + \frac{t^4}{4} + \frac{1}{2}t - \frac{1}{2}$.

30. Arrange in ascending powers :

(i) $8p + 3 + p^3 - p$; (ii) $\frac{c^2}{2} + 1\frac{1}{2} - \frac{c^2}{4} + 2c$.

31. Write down (i) the coefficient of x^2, (ii) the constant term, (iii) the term of degree 3 in

(*a*) $4 + 2x^2 + 3x^3$; (*b*) $x^6 + x^2$;
(*c*) $2x^3 - x + 5$; (*d*) $3x^4 + 2x^3 + 7x + 4$.

32. Write down (i) the term of highest degree, (ii) the coefficient of p, (iii) the constant term in

(*a*) $12p^3 + 2p^4 + 7p + 2\frac{1}{2}$; (*b*) $p + 100p^2 + 3p^3$;
(*c*) $12p^2 + 2p^4 + 5$; (*d*) $\frac{1}{2}p^3 - 8p^2$.

33. Write (i) in ascending powers, (ii) in descending powers

(*a*) $4y^4 + 3 - 2y^2$; (*b*) $10t^2 + t^3 + 2t$;
(*c*) $s^4 + 1 + 20s$; (*d*) $3z + z^3 - 2z + 3 + 5z^2$.

Multiplication and Division

Example 7. Multiply x^2 by x^3.

$$x^2 = x \times x \text{ and } x^3 = x \times x \times x ;$$

$$\therefore x^2 \times x^3 = x \times x \times x \times x \times x = \mathbf{x^5}.$$

Example 8. Multiply $3x$ by $4y$.

$$3x \times 4y = 3 \times 4 \times x \times y = 12 \times x \times y = \mathbf{12xy}.$$

Example 9. Multiply $12x^3$ by $\frac{2}{3}xy$.

$$12x^3 = 12 \times x^3 \text{ and } \tfrac{2}{3}xy = \tfrac{2}{3} \times x \times y.$$

$$\therefore 12x^3 \times \tfrac{2}{3}xy = 12 \times x^3 \times \tfrac{2}{3} \times x \times y = 12 \times \tfrac{2}{3} \times x^3 \times x \times y$$
$$= \mathbf{8x^4y}.$$

Example 10. Divide a^6 by a^2.

$$a^6 \div a^2 = \frac{a \times a \times a \times a \times a \times a}{a \times a} = a \times a \times a \times a$$
$$= \mathbf{a^4}.$$

Example 11. Divide $12a^3bc$ by $9a^2b$.

$$12a^3bc \div 9a^2b = \frac{12 \times a \times a \times a \times b \times c}{9 \times a \times a \times b} = \frac{4}{3} \times a \times c$$

$$= \mathbf{\frac{4ac}{3}}.$$

Example 12. What is the square of a^4?

$$(a^4)^2 = a^4 \times a^4 = \mathbf{a^8}.$$

Example 13. Find a value of $\sqrt{x^6}$.

$$x^3 \times x^3 = x^6\text{; } \therefore \text{ the square of } x^3 \text{ is } x^6\text{;}$$

$$\therefore \textit{ one } \text{value of } \sqrt{x^6} \text{ is } \mathbf{x^3}.$$

We shall see later that every number has two square roots.

EXERCISE IV. d

Simplify the expressions, Nos. 1-12, giving the reasons in full:

1. $a^3 \times a^3$.	2. $b \times b^4$.	3. $c^5 \div c^2$.	4. $d^4 \div d$.
5. $p^4 \times p^2$.	6. $t^6 \div t^3$.	7. $(x^3)^2$.	8. $(y^2)^3$.
9. $5x \times 2y$.	10. $3x^2 \times 2x$.	11. $\frac{1}{2}(2r)^3$.	12. $3s^3 \times 3s^3$.

Simplify the following:

13. $a^2 \times a^4$.	14. $b^6 \times b^2$.	15. $c^6 \div c^2$.	16. $d^8 \div d^4$.
17. $r \times r^3$.	18. $s^5 \div s$.	19. $t^4 \times t^4$.	20. $x^3 \div x^3$.
21. $4x^2 \times 2x$.	22. $6y^2 \div 2y$.	23. $z^2 \times 3z$.	24. $2ab \times a$.
25. $10a^4 \div 2a^2$.	26. $3b \times 3b^3$.	27. $4c \div c$.	28. $2t^2 \times 4t^4$.
29. $4d \times \frac{d}{2}$.	30. $6 \times \frac{e}{4}$.	31. $\frac{a^2}{2} \times \frac{a^3}{3}$.	32. $\frac{2s}{3} \times \frac{3s}{2}$.
33. $4d^2 \div \frac{d}{2}$.	34. $10p^2 \div 2p^2$.	35. $8r^2 \div 6$.	36. $12xy^2 \div 3y$.
37. $ab \times ac$.	38. $2b \times bc$.	39. $2c^2d \div cd$.	40. $4m^4 \div 2$.
41. $(3d^3)^2$.	42. $(rs^4)^2$.	43. $3pq \times 2pr$.	44. $2r^2s \div 3s$.
45. $2t^3 \times 3t^3$.	46. $rt^2 \div \frac{1}{2}rt$.	47. $6R^3 \div \frac{1}{2}R$.	48. $\sqrt{z^8}$.

49. $4b^3c \div \frac{1}{2}bc$.	50. $3rs \times 2rst$.	51. $x^2 \times 2yz$.
52. $6a^3b^2c^2 \div 2abc$.	53. $6c^6d^2 \div 4c^3$.	54. $(4e^3f)^2$.
55. $(\frac{1}{2}rs^2)^3$.	56. $(4pq^3r)^2$.	57. $(3r^2s^3t^4)^2$.
58. $(x^3)^3 \div x^3$.	59. $(2x)^2 + (3x)^2$	60. $(6y)^3 \div (2y)^3$.
61. $\frac{a^4 \times a^4}{a^8}$.	62. $\frac{b \times b^2 \times b^4}{b^3}$.	63. $\frac{6cd \times 2cd^2}{10c}$.
64. $\frac{ab \times ac}{bc}$.	65. $2rs \times (3r)^2$.	66. $\sqrt{(16t^{16})}$.
67. $6c^2d \times \frac{1}{2}cde$.	68. $(3s)^3 - 2(2s)^3$.	69. $(2pq)^3 \div pq^3$.
70. $\frac{r^6}{r^3} + 3r^3$.	71. $t^6 \times t^4 \div t^2$.	72. $2n^2 \times 6nk^3 \div 3n$.

[*Note. For additional drill-examples, see Exercise E.P.* 5, *p.* 137.]

H.C.F. and L.C.M.

Factors and Multiples. Since $10ab^2c = 5ab \times 2bc$, we say that $2bc$ is a *factor* of $10ab^2c$ and that $10ab^2c$ is a *multiple* of $2bc$.

Common Factors. Consider the two expressions, $10ab^2c$, $12a^2b^2$.

$$10ab^2c = 2 \times 5 \times a \times b \times b \times c\ ;\ 12a^2b^2 = 2 \times 2 \times 3 \times a \times a \times b \times b.$$

There are several different expressions which are factors of both these expressions, *i.e. common* factors of $10ab^2c$, $12a^2b^2$.

$$2,\ a,\ b,\ b^2,\ 2a,\ 2b,\ 2b^2,\ ab,\ ab^2,\ 2ab,\ 2ab^2.$$

But each of these is a factor of the last one, $\mathbf{2ab^2}$.

We therefore call $\mathbf{2ab^2}$ the Highest Common Factor or the H.C.F. of $10ab^2c$, $12a^2b^2$.

This method for finding the H.C.F. is the same as the method used in Arithmetic (prime factors).

Common Multiples. Consider the two expressions, $10ab^2c$, $12a^2b^2$.

$10ab^2c \times 12a^2b^2 = 120a^3b^4c$; $\therefore$ $120a^3b^4c$ is a *common* multiple of $10ab^2c$, $12a^2b^2$. There are of course an unlimited number of common multiples, *e.g.* $240a^4b^4c^4d$ is a common multiple.

Since $10ab^2c = 2 \cdot 5 \cdot a \cdot b^2 \cdot c$ and $12a^2b^2 = 2^2 \cdot 3 \cdot a^2 \cdot b^2$, the smallest common multiple is $2^2 \cdot 3 \cdot 5 \cdot a^2 \cdot b^2 \cdot c = \mathbf{60a^2b^2c}$.

Every other common multiple is itself a multiple of $60a^2b^2c$.

We therefore call $\mathbf{60a^2b^2c}$ the Least Common Multiple or the **L.C.M.** of $10ab^2c$, $12a^2b^2$.

This method for finding the L.C.M. is the same as the method used in Arithmetic (prime factors).

EXERCISE IV. e

1. Is 12 a multiple of (i) 4, (ii) 48 ?
Is $6ab$ a multiple of (i) $2a$, (ii) $6a^2b^2$?

2. Is 18 a multiple of (i) 180, (ii) 6 ?
Is a^3 a multiple of (i) a^2, (ii) $2a^6$?

3. Is 24 a factor of (i) 12, (ii) 48 ?
Is $6xy$ a factor of (i) $3x$, (ii) $18xy^2$?

4. Is 24 a common multiple of (i) 2 and 3, (ii) 48 and 240 ?
Is $6x^2y$ a common multiple of (i) $3x$, $2xy$, (ii) $6x^2y^2$, $12x^3y^3$?

5. Is 12 a common factor of (i) 3 and 4, (ii) 24 and 36 ?
Is $6ab$ a common factor of (i) $2a$, $3b$, (ii) $12a^2b$, $18ab^2$?

6. Is $10x^4y^3$ a multiple of $5x^3y$? Simplify $\frac{10x^4y^3}{5x^3y}$.

7. Is $3b^2c$ a factor of $6bc^2$?

8. Is $2y^2z$ a common factor of $6y^3z$ and $4y^2z^3$?

9. Is $12x^2y^3$ a common multiple of any of the following pairs : (i) $3x^2y$, $4xy^2$; (ii) $12x^2$, y^3 ; (iii) $10x$, $4y$; (iv) $3x^2$, y^4 ?

10. Which of the pairs in No. 9 have a common factor, other than unity, and what is it ?

Find the following :

11. L.C.M. of $3x$, $2xy$.
12. H.C.F. of $3a^2$, $2ab^2$.
13. H.C.F. of $10a^3$, $15a^2b$.
14. L.C.M. of $4xy$, $6y^2$.
15. L.C.M. of $4r^2s$, $5rs^2$.
16. H.C.F. of $2pq$, pqr^2.
17. L.C.M. of a, bca^2.
18. H.C.F. of $2ab$, $3c$.
19. L.C.M. of $3a^2b^2$, $2a^3b^3$.
20. H.C.F. of $2a^2$, $6ab$, $4ac$.
21. L.C.M. of $6x^2$, $3xy$, $6y^2$.
22. L.C.M. of $10x^2$, $15x^3$.
23. H.C.F. and L.C.M. of 24, $3x$, $3xy$, $6xz$.
24. H.C.F. and L.C.M. of $12a^3b$, $9abc$, $15a^2b^3$, $24ab^2x$.

[*Note. For additional drill-examples, see Exercise E.P. 6, p. 138.*]

Fractions

Expressions involving Fractions

Example 14. (i) On a farm of 480 acres, there are 190 acres of arable land and the rest is pasture. What fraction of the farm-land is pasture ?

(ii) On a farm of n acres, there are p acres of arable land and the rest is pasture. What fraction of the farm-land is pasture ?

(i) Out of 480 ac., there are 190 ac. of arable land;

$\therefore$ there are $(480 - 190)$ ac. $= 290$ ac. pasture;

$\therefore$ the fraction of the farm under grass is $\frac{290}{480} = \frac{29}{48}$.

(ii) Out of n ac., there are p ac. of arable land ;

$\therefore$ there are $(n - p)$ ac. pasture ;

$\therefore$ the fraction of the farm under grass is $\frac{n-p}{n}$.

EXERCISE IV. f

1. What fraction is (i) 4 pence of 6 pence, (ii) c pence of n pence, (iii) p pence of s shillings, (iv) r shillings of £t ?

2. Express (i) 8 inches in feet, (ii) l inches in feet, (iii) m inches in yards, (iv) k yards in feet, (v) p feet in yards.

3. Express p shillings q pence (i) in pence, (ii) in s., (iii) in £.

4. (i) A school contains 400 pupils; of these, 240 are boys; what fraction of the school are girls?

(ii) A school contains p pupils; of these, b are boys; what fraction of the school are girls?

(iii) In a school, there are b boys and g girls; what fraction of the school are boys?

5. From a stick l inches long, a portion p inches long is cut off. What fraction of the stick remains?

6. After the nth day of January, what fraction of January remains?

7. A boy sleeps k hours a day; what fraction of the day is he awake?

8. After motoring s miles, I have gone $\frac{1}{n}$th of the total distance. What is the total distance? How much further have I to go?

9. After $\frac{2}{3}$ of a chest of tea has been used, p lb. of tea remain. What did the chest hold originally?

10. The Sun rises at x a.m. and sets at y p.m.; how many hours long is the day? What fraction is this of 24 hours?

11. Find with the data of Fig. 73, the values of

$$\text{(i) } \frac{AB}{BC}, \quad \text{(ii) } \frac{BC}{AC}, \quad \text{(iii) } \frac{AC}{AB}.$$

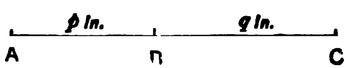


FIG. 73.

12. How many $1\frac{1}{2}$d. stamps can be bought for (i) 1s., (ii) ps.? What fraction is $1\frac{1}{2}$d. of q shillings?

13. Taking 8 km. as equal to 5 miles, express (i) l km. in miles, (ii) s miles in km.

14. Sweets are sold at 5d. for 2 oz. What is the cost of n oz.? What amount is obtained for k shillings?

15. What fraction is (i) 5 inches of 5 feet, (ii) l inches of l feet, (iii) $3l$ inches of l feet?

16. A shed l ft. long, b ft. wide, see Fig. 74, is built in the corner of a rectangular enclosure x ft. long, y ft. wide. What fraction of the total area does the shed occupy? What is the value of this fraction if

(i) $x = 2l$, $y = 2b$, (ii) $x = 2l$, $y = 3b$?

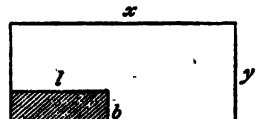


FIG. 74.

17. (i) If a man walks 4 miles an hour, how long does he take to walk p miles, $(p+q)$ miles ?

(ii) If a man cycles 12 miles an hour, how long does he take to cycle $3p$ miles ? Simplify $\frac{3p}{12}$; $\frac{5p}{20}$.

18. Simplify (i) $\frac{70}{80}$; (ii) $\frac{7p}{8p}$; (iii) $\frac{6p}{8}$; (iv) $\frac{9p}{12q}$.

19. Simplify (i) $\frac{1}{5}+\frac{3}{4}$; (ii) $\frac{p}{5}+\frac{p}{4}$; (iii) $\frac{a}{5}+\frac{b}{4}$.

20. Simplify (i) $\frac{3}{4}-\frac{1}{5}$; (ii) $\frac{p}{4}-\frac{p}{5}$; (iii) $\frac{a}{4}-\frac{b}{5}$.

21. If in Fig. 73 AB is x yards, BC is y yards, express the lengths of AB and BC also in feet and in inches. Then express the fraction $\frac{\text{AB}}{\text{BC}}$ in three different ways.

22. How many parts, each $\frac{2}{3}$ inch long, can be cut off along a line (i) 4 inches long, (ii) b inches long ? Simplify $b \div \frac{2}{3}$.

23. How many $\frac{3}{4}$ lb. packets can be made up from n lb. of tea ?

24. The price of coal rises from xs. per ton to ys. per ton ; how many fewer tons can now be bought for £p ?

25. A tap can fill a bath in t minutes. What fraction of the bath is filled in (i) 3 minutes, (ii) half a minute ?

26. One tap can fill a bath in p minutes and another tap can fill it in q minutes. What fraction of the bath is filled by both taps together, (i) in 1 minute, (ii) in 4 minutes ?

27. Fine parallel lines are engraved on a bar at intervals of $\frac{1}{t}$ of an inch. There are n lines in all. What is the distance between the first line and the last ? [Invent a numerical example.]

28. A packet of paper 1 inch high contains n sheets. How thick is each sheet ?

29. A packet of paper $\frac{1}{2}$ inch high contains $2p$ sheets. How thick is each sheet ?

30. (i) A sheet of paper is $\frac{1}{100}$ inch thick. How many sheets are there in a pile 3 inches high ?

(ii) A sheet of paper is $\frac{1}{n}$ inch thick. How many sheets are there in a pile h inches high ?

Simplification of Fractions

Example 15. Simplify $\frac{6a^2b^2}{15ab^3}$.

Divide the numerator and denominator by every common factor.

$$\frac{6a^2b^2}{15ab^3}=\frac{\overset{2}{\cancel{6}}\;\overset{a}{\cancel{a^2}}\;\overset{1}{\cancel{b^2}}}{\underset{5}{\cancel{15}}\;\underset{1}{\cancel{a}}\;\underset{b}{\cancel{b^3}}}=\frac{2a}{5b}.$$

Or, proceed as follows :

The H.C.F. of $6a^2b^2$ and $15ab^3$ is $3ab^2$.

Divide the numerator and denominator by $3ab^2$.

$$\therefore\ \frac{6a^2b^2}{15ab^3}=\frac{3ab^2\times 2a}{3ab^2\times 5b}=\frac{2a}{5b}.$$

EXERCISE IV. g

Simplify the following :

1. $\frac{3a}{3c}$. 2. $\frac{3a}{5a}$. 3. $\frac{6c}{6}$. 4. $\frac{3d}{d}$. 5. $\frac{2ab}{3ac}$.

6. $\frac{2a^2}{5a}$. 7. $\frac{2bc}{c^2}$. 8. $\frac{2p}{2p}$. 9. $\frac{r^2}{2r}$. 10. $\frac{3s^2}{6st}$.

11. $\frac{4bc}{6cd}$. 12. $\frac{8de}{6d^2c}$. 13. $\frac{r}{rs}$. 14. $\frac{st}{st}$. 15. $\frac{2s^2t^2}{st}$.

16. $\frac{12a}{12ab}$. 17. $\frac{4a^4}{6a^6}$. 18. $\frac{a^2xy}{ayz}$. 19. $\frac{xy^2}{y^2x}$. 20. $\frac{3cd}{6c^2d^2}$.

21. $\frac{4a^6b}{2a^3c}$. 22. $\frac{a^4}{a^8}$. 23. $\frac{12x^2y^2}{8xyz}$. 24. $\frac{a^6c^6}{a^3c^2}$. 25. $\frac{a^3b^3c^3}{abc}$

26. $\frac{3pq}{p^3}$. 27. $\frac{x^2.(xy)}{(xy)^2}$. 28. $\frac{3ab^2}{6b^2a^2}$. 29. $\frac{x^2y^2}{2xy}$.

30. $\frac{4a^4b^2c^2}{2a^2bc}$. 31. $\frac{b^{10}}{2b^5}$. 32. $\frac{(3xy)^2}{3xy^2}$. 33. $\frac{a^2b\,.\,ac^3}{ca\,.\,ba}$

Example 16. Express $\frac{3x}{y}$ in the form $\frac{?}{2y^2z}$.

$$2y^2z=y\times 2yz\ ;\ \therefore\ \frac{3x}{y}=\frac{3x\times 2yz}{y\times 2yz}=\frac{6xyz}{2y^2z}.$$

Example 17. Simplify $\frac{2p}{3}-\frac{p}{4}+\frac{5p}{6}$.

The L.C.M. of 3, 4, 6 is 12.

$$\frac{2p}{3}-\frac{p}{4}+\frac{5p}{6}=\frac{8p}{12}-\frac{3p}{12}+\frac{10p}{12}=\frac{8p-3p+10p}{12}$$
$$=\frac{15p}{12}=\frac{5p}{4}.$$

Example 18. Simplify $\frac{7}{10a}+\frac{2}{15a}$.

The L.C.M. of $10a$, $15a$ is $30a$.

$$\frac{7}{10a}+\frac{2}{15a}=\frac{21}{30a}+\frac{4}{30a}=\frac{21+4}{30a}$$
$$=\frac{25}{30a}=\frac{5}{6a}.$$

Example 19. Simplify $\frac{b}{6c}-\frac{c}{4b}$.

The L.C.M. of the denominators $6c$, $4b$ is $12bc$.

$$\frac{b}{6c}-\frac{c}{4b}=\frac{b\times 2b}{6c\times 2b}-\frac{c\times 3c}{4b\times 3c}=\frac{2b^2}{12bc}-\frac{3c^2}{12bc}$$
$$=\frac{2b^2-3c^2}{12bc}.$$

Example 20. Express as a single fraction $\frac{r^2}{s}-t$.

Since $t=\frac{t}{1}$, the L.C.M. of the denominators s, 1 is s.

$$\therefore\ \frac{r^2}{s}-t=\frac{r^2}{s}-\frac{ts}{s}=\frac{r^2-ts}{s}.$$

Note. As soon as the method is understood, the two intermediate steps in Example 19 should not be written down.

EXERCISE IV. h

Copy and complete the following, Nos. 1-14:

1. $\frac{3}{4}=\frac{}{12}=\frac{}{28}=\frac{}{100}$.

2. $\frac{a}{b}=\frac{}{2b}=\frac{}{bc}=\frac{}{b^2}$.

3. $\frac{8}{5}=\frac{}{25}=\frac{24}{}=\frac{64}{}$.

4. $\frac{c}{d}=\frac{}{4d}=\frac{5c}{}=\frac{ac}{}$.

5. $\frac{3r}{4s}=\frac{}{8s}=\frac{}{8st}=\frac{3r^2}{}$.

6. $b=\frac{}{3}=\frac{}{a}=\frac{}{b}$.

7. $\frac{l}{m}=\frac{}{m^2}=\frac{}{3lm}=\frac{kl}{}$.

8. $\frac{2t^2}{4tz}=\frac{}{2z}=\frac{3t}{}=\frac{t^3}{}$.

9. $\frac{x}{3y}=\frac{x^2}{}$.

10. $\frac{1}{yz}=\frac{y}{}$.

11. $\frac{1}{e^2}=\frac{}{e^6}$.

12. $f=\frac{}{2g}$.

13. $h^2=\frac{3h^3}{}$.

14. $m^3=\frac{}{m^3}$

Simplify the following:

15. $\frac{2}{3}+\frac{1}{7}$; $\frac{a}{3}+\frac{a}{7}$; $\frac{2a}{3}+\frac{a}{7}$; $\frac{5a}{3}+\frac{2a}{7}$.

$\frac{3}{5}-\frac{2}{7}$; $\frac{b}{5}-\frac{b}{7}$; $\frac{3b}{5}-\frac{2b}{7}$; $\frac{5b}{5}-\frac{4b}{7}$.

17. $\frac{3}{8}+\frac{4}{8}$; $\frac{3}{n}+\frac{4}{n}$; $\frac{3}{5n}+\frac{4}{5n}$; $\frac{1}{n}+\frac{2}{5n}$.

18. $\frac{5}{9}-\frac{2}{9}$; $\frac{5}{d}-\frac{2}{d}$; $\frac{5}{3d}-\frac{2}{3d}$; $\frac{5}{2d}-\frac{4}{3d}$.

19. $\frac{3}{4}+\frac{7}{10}$; $\frac{3}{2c}+\frac{7}{5c}$; $\frac{3}{2c}-\frac{7}{5c}$; $\frac{6}{2c}-\frac{10}{5c}$.

20. $1+\frac{4}{7}$; $1+\frac{r}{s}$; $1+\frac{r}{2s}$; $\frac{r}{2s}+1$.

21. $2-\frac{7}{5}$; $2-\frac{x}{y}$; $1-\frac{x}{2y}$; $n-\frac{x}{3y}$.

22. $\frac{1}{3t}+\frac{1}{6t}$.

23. $\frac{1}{g}-\frac{1}{4g}$.

24. $\frac{b}{2c}-\frac{b}{6c}$.

25. $\frac{m}{n}-1$.

26. $\frac{1}{m}+\frac{1}{p}$.

27. $\frac{1}{2u}+\frac{1}{3v}$.

28. $\frac{r}{s}+\frac{s}{r}$.

29. $\frac{2r}{s}-t$.

30. $1-\frac{1}{ab}$.

31. $\frac{1}{c}-\frac{c}{2}$.

32. $\frac{3}{5d}-\frac{1}{10d}$.

33. $\frac{2x}{3y}-\frac{x}{6y}$.

34. $\frac{e}{f}-\frac{e^2}{f^2}$.

35. $\frac{r}{4s}+\frac{s}{6r}$.

36. $t-\frac{2}{5}t$.

37. $\frac{a}{4ab}+\frac{c}{6bc}$.

38. $\frac{x}{2y}+\frac{y}{2x}-2$.

39. $\frac{3c}{2d}+\frac{2c}{3d}+\frac{5c}{6d}$.

40. $\frac{b}{c}-\frac{c}{d}+\frac{d}{b}$.

41. $\frac{p}{qr}+\frac{q}{rp}-\frac{r}{pq}$.

42. $\frac{ab}{bc}-\frac{ab}{ac}-1$.

43. Add $\frac{p}{q}$ to $\frac{p^2}{q^2}$.

44. Subtract 1 from 2

45. Subtract $\frac{1}{15n}$ from $\frac{2}{5n}$.

46. Add $\frac{u}{3v}$ to $\frac{u^2}{6uv}$.

47. Add $\frac{r}{4s}$ to $\frac{r}{12s}$.

48. Subtract $\frac{1}{z}$ from z.

49. $\frac{7}{20}\times 60$; $\frac{a}{2b}\times 6b$.

50. $\frac{15}{14}\div 3$; $\frac{3d}{2c}\div 3$.

51. $\frac{5}{11}\times 2$; $\frac{e}{f}\times 2$.

52. $1\div\frac{3}{4}$; $1\div\frac{x}{2y}$.

53. $p\times\frac{q}{r}$.

54. $p\div\frac{q}{r}$.

55. $2b\times\frac{c^2}{bd}$.

56. $\frac{l}{m}\div\frac{r}{t}$.

57. $\frac{u}{v}\times\frac{v}{u}$.

58. $\frac{y^2}{z}\times\frac{z^2}{y}$.

59. $a\div\frac{a^2}{b^2}$.

60. $\frac{x}{y}\times\frac{y}{z}\div\frac{x}{z}$.

61. $p\div\frac{1}{q}$.

62. $\frac{1}{6c^2}\div\frac{1}{4cd}$.

63. $3m\times\frac{3n}{m^2}$.

64. $ab^2\times\frac{1}{b^2a}$.

65. $\frac{2ab}{15c^2}\times\frac{5bc}{4a^2}$.

66. $\frac{1}{d^3}\div d^3$.

67. $\frac{a^6}{b^2}\times\frac{b^6}{a^2}$.

68. $\frac{4a^2}{6bc}\div\frac{3ab}{5ac}$.

69. $\left(\frac{u}{3v}\right)^2\times 3v$.

70. $\frac{ab}{c}\div\frac{b}{ca}$.

[*Note. For additional drill-examples, see Exercise E.P.* 7, *p.* 138.]

Miscellaneous Examples

EXERCISE IV. j

1. How many inches are there in (i) $\frac{s}{3}$ feet, (ii) $\frac{t}{4}$ yards ?

2. How many yards are there in (i) $4\frac{1}{2}l$ feet, (ii) $8n$ inches ?

3. What is the length in yards of a train which has n coaches, if each coach is c feet long and the engine is $6p$ feet long ?

4. A clerk writes n letters an hour, how many minutes does he take to write $\frac{r}{5}$ letters ?

5. A rug is 6 feet wide, l feet long ; what is its area in (i) sq. ft., (ii) sq. yd. ?

6. A man smokes $\frac{1}{4}$ lb. of tobacco a week ; how long does W lb. of tobacco last him ?

7. The length of fence for a square enclosure is $2p$ yd. ; what is the area enclosed in (i) sq. yd., (ii) sq. ft. ?

8. Fig. 75 represents the floor of a hall. What is its floor area in terms of l, if the longer side is $3l$ feet ?

9. Fig. 75 represents the floor of a hall. Find in sq. feet its floor area in terms of p, if its perimeter is p yards.

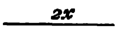


FIG. 75.

10. In Fig. 76, what fraction is the shortest side of the perimeter of the triangle ?

11. In Fig. 76, the longest side is $2d$ inches. Find in terms of d the perimeter of the triangle.

12. In Fig. 76, the perimeter is p feet ; find in terms of p, the longest side.

13. What relation connects the areas of the three squares whose sides are those of the triangle in Fig. 76 ?

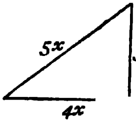


FIG. 76.

14. A pencil costs $\frac{3p}{2}$ pence ; how many pencils can I buy for half-a-crown ?

15. I bicycle at $\frac{1}{2}v$ miles an hour ; how far do I go in 40 minutes ?

16. The average speed of a train is $10u$ miles an hour ; how long does it take to go 50 miles ? How long would it take if the speed was 10 miles an hour less ?

17. Take the number N and from half of it subtract one-third of it. By what must the result be multiplied to give N ?

18. From a square sheet of cardboard of side $3t$ inches a square of side $2t$ inches is cut away. What area remains ?

19. A train has $11n$ compartments with 8 seats each and $2n$ compartments with 6 seats each ; how many seats are there on the train ?

20. With the data of No. 19, find the number of such trains required for 2000 passengers if each has a seat.

21. Fig. 77 shows the dimensions of a brick in inches. What is the sum of its edges in feet ?

22. Find in terms of t the total area of the surface of the brick in Fig. 77, if $h=\frac{t}{3}$; the units of the given dimensions being inches.

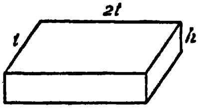


FIG. 77.

23. A handkerchief costs k pence and a pair of socks costs one shilling more. What is the total cost of 8 handkerchiefs and 4 pairs of socks (i) in pence, (ii) in shillings ?

24. It takes x men t days to repair a certain road ; how long should it take $2x$ men ?

25. A tank is $4p$ ft. long, $2p$ ft. wide and contains water to a depth of $3p$ ft.; what is the area of the wetted surface ?

26. How many tiles, measuring 8 in. by 6 in., are required for the floor of a hall $2b$ ft. long, b ft. broad ?

27. A photograph l in. long, w in. wide is printed on a sheet of paper $\frac{5l}{4}$ in. long, $2w$ in. wide. What area of the paper is not used ?

28. A certain milestone on the Winchester-Southampton road reads $(n+1)$ miles to Winchester, $2n$ miles to Southampton. How far is Winchester from Southampton ? What is n if this distance is 13 miles ?

29. A shopkeeper would make a profit of £P by selling a table for £S. For what must he sell it to make a profit of (i) £(2P), (ii) £($\frac{1}{2}$P), (iii) £R.

30. At a shooting gallery, you pay 2d. if you miss and receive 6d. if you hit the target. What is the result if your score is $\frac{h}{3}$ hits and $\frac{m}{2}$ misses ?

[*Note. For additional examples, see Appendix, Ex. S. 5, p. 285. For a revision exercise on Ch. III-IV, see Appendix, Ex. R. 2, p. 259.*]

TEST PAPERS, A. 9-15

A. 9

1. (i) Add $6ab - 3a - 2b$ to $6a + 4b - 2ab$.

(ii) For what value of W are the expressions $\frac{W}{2} + 1$ and $\frac{2W}{3} - 1$ equal ?

2. Copy and complete the following table, if $4x + 3y = 10$;

$x =$ 0		1		2	
$y =$	0		1		3

3. A car uses a gallon of petrol every 18 miles. How far can the car run on k gallons ?
The car travels at v m.p.h.; how much petrol is used in half an hour ?

4. The relation between the thickness T inches and the diameter D inches of a steam engine cylinder is

$$T = \tfrac{1}{8}\sqrt{D} + 0{\cdot}015D.$$

What is T when $D = 16$?

What is the increase in T if D is now increased by 9?

5. The base of a prism is an n-sided figure. What is the number of (i) its edges, (ii) its corners, (iii) its faces, including the two ends?

What is n if the number of edges is $2\tfrac{1}{2}$ times the total number of faces?

A. 10

1. (i) Square $2x$ and halve the result.

(ii) If $r = \tfrac{3}{4}R$ and if $r = 9$, what is the value of $R - r$?

2. (i) Find the H.C.F. of $4abc^2$, $6a^2bc$, $8a^3c^3$.

(ii) Simplify $\dfrac{10xyz^2}{6x^2yz}$.

3. Solve the equations,

(i) $0{\cdot}6l = 9$; (ii) $\dfrac{n}{3} + \dfrac{n}{5} = 0$; (iii) $3(a + 4) + 2(a - 2) = 20$.

4. Houses along a side road are built in pairs (semi-detached) with s yards between each pair; the front width of each pair is l feet. What length of road, in yards, is needed for 12 houses?

5. A man walks in the direction $x°$ E. of N. along AB, see Fig. 78, he turns clockwise through $\left(\dfrac{x}{2} + 5\right)$ degrees at B and walks along BC, and then turns anti-clockwise through $(2x - 15)$ degrees at C; he is now walking due North; find x.

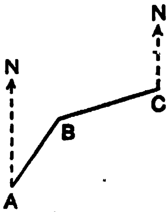


FIG. 78.

A. 11

1. (i) Divide $2p$ by $3p$; subtract the result from p.

(ii) If $u = 1\tfrac{2}{3}$ and $v = 2$, find the value of $\dfrac{1}{u} + \dfrac{1}{v}$.

2. (i) A lift is licensed to carry n persons, on the assumption that the average weight of a person is less than 16 stone. What is regarded as the maximum safe load in tons?

(ii) How many eggs at 2 for 3d. can I buy with k florins?

3. Solve the equations,

(i) $\dfrac{h}{3} + 12 = h$; (ii) $3(2y - 1) + 5(3y + 2) = 8(3y - 1)$.

4. A is 6 years old and B is 29; in how many years' time will A be just half as old as B?

5. The stretched length of a spiral spring is $(21 + 2\frac{1}{2}W)$ inches, when it is carrying a load of W lb. What is the natural (unstretched) length of the spring ? What load is required to stretch it to twice its natural length ?

A. 12

1. If $p=3$, $q=5$, $r=7$, find the values of

   (i) $p+r(p+q)$; (ii) $(p+r)q+r$; (iii) $(p+q)\div(p+r)$.

2. (i) Add $3a-b+2c$ to $3c-b-2a$; (ii) express the sum in terms of b if $a+1=b=c-1$.

3. (i) Solve the equation, $\frac{1}{5}(z-3)=\frac{1}{2}$.

   (ii) When the day is t hours long, the night is $\frac{3}{5}t$ hours long. What is t ?

4. A hawser, C inches in circumference, breaks under a load of W tons, where $W=\frac{2}{5}C^2$; the working load is not allowed to exceed one-sixth of the breaking load. What is the maximum working load for a hawser whose circumference is 3 inches ?

5. Share £1.10s. between A, B, C so that A gets 6 shillings more than B, and C gets $\frac{3}{7}$ of what A and B together got.

A. 13

1. Simplify, and write in ascending powers of x,

$$4x^4+5x^3-3x+7+4x-4x^3-x-1.$$

What is (i) the coefficient of x^3, (ii) the coefficient of x, (iii) the constant term ?

2. Simplify (i) $\dfrac{1}{15n}+\dfrac{1}{30n}$; (ii) $2t^2\times 3t^3 \div 5t^5$.

3. A knife costs p pence and a fork costs half as much again ; what is the cost in shillings of a dozen forks ?

4. Solve the equations,

$$\text{(i) } \frac{1}{u}+\frac{1}{5}=\frac{1}{3}\,; \qquad \text{(ii) } n+\frac{5n}{7}=0{\cdot}36.$$

Prove that the equation $x^2(11-x)=36(x-1)$ is satisfied by $x=2$ and by $x=3$ and by $x=6$.

5. The length of wire required to make the rectangular grid

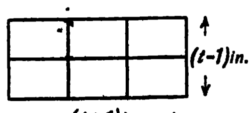


FIG. 79.

in Fig. 79 is 20 inches. Find the length and breadth of the grid.

A. 14

1. (i) If $R = 3r$, simplify $\dfrac{R^2 - r^2}{R^2 + r^2}$.

(ii) If $\dfrac{x}{2} = \dfrac{3}{4} = \dfrac{5}{y}$, find the value of $3y(x - 1)$.

2. A man stays 10 days at a hotel; he is charged 15s. a day for the first $2n$ days and 12s. 6d. a day for the rest of the time. What is his total bill (i) in shillings, (ii) in £? Assume $n < 5$.

3. (i) Add $\dfrac{r}{4s}$ to $\dfrac{r^2}{12rs}$; (ii) Subtract 1 from $\dfrac{a+b}{2b}$.

4. Solve the equations,

(i) $l - 0{\cdot}3l = 14$; (ii) $\frac{3}{5}(W - 1) = \frac{1}{4}(W + 5)$.

5. A has three times as much money as B. If A gives B four shillings, he will then have just twice as much as B; how much have they between them?

How much must B give A in order that A may have five times as much as B?

A. 15

1. If $r = 5$, $s = 0$, $t = 3$, find the values of

(i) $(r + s)t$; (ii) $2s(r + t)$; (iii) $(r^2 - t^2) - (r - t)^2$.

2. A rectangle is $7k$ inches long, $3k$ inches wide; find the area of a square whose perimeter is the same as that of the rectangle. Which has the larger area, the rectangle or the square, and by how much?

3. A workman is paid 2s. an hour ordinary time and 4s. an hour overtime. He receives £P for a 50-hour week. How many of the 50 hours are counted as overtime? What is the least possible value of P?

4. (i) Solve the equation, $\dfrac{1}{t} + \dfrac{1}{2t} = \dfrac{1}{6}$.

(ii) If $\frac{3}{4}(n + 1)$ and $\frac{3}{5}n + 1$ are equal, prove that each of them must equal n.

5. A starts at a salary of £250 a year and receives an annual increase of £15 a year. B starts at £320 a year and receives an annual increase of £10 a year. After how many years does A receive a larger salary than B?

[*For additional test papers on Ch. I-IV, see Appendix, P.* 1-5, *p.* 312.]

CHAPTER V

THE A B C OF GRAPHS

Graphs on Plain Paper

The following example is intended for oral work.

Example 1. A motor-car is fitted with a gauge which shows the number of gallons of petrol in the tank. When full, the tank holds 5 gallons. A motorist starts out at 10 a.m. and notes the readings on the gauge at hourly intervals. The result is shown in Fig. 80.

Petrol gauge readings.

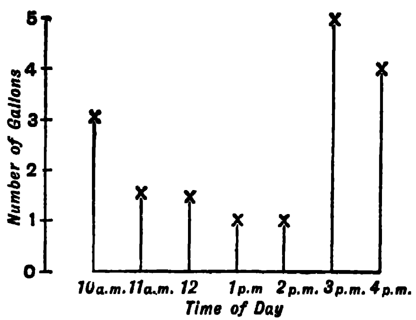


FIG. 80.

EXERCISE V. a (Oral)

Use the data and figure of Example 1, above, to answer the following questions:

1. What is the length in inches of the line which represents (i) 2 gallons, (ii) 5 gallons ?

2. How much petrol is represented by an upright line of length (i) 1 inch, (ii) ½ inch, (iii) 1·5 inches ?

3. How much petrol did he start with ? How much had he at 1 p.m. and at 4 p.m. ?

4. During what time is it probable that the car was not running ?

5. About what time did he fill up the tank ?

6. If the car averages 24 miles to the gallon, estimate the distances travelled in successive hours.

7. Can you give a meaning to an upright inserted midway between the first and second uprights ; if so, what meaning ?

8. Can you insert with fair accuracy an upright for 2.30 p.m. ?

EXERCISE V. b

Draw on *plain* paper, as in Fig. 80, diagrams to represent the records tabulated below. State in each case (i) whether any meaning can be given to intermediate upright lines, (ii) whether interpolation is possible with fair accuracy without further data.

Give each diagram a title, and write along each axis what that axis represents and, where suitable, how it is graduated.

1. The distribution over the world of the chief languages :

Language.	English.	German.	Russian.	French.	Spanish.
Number who speak it, in millions -	160	100	100	70	50

2. The average diameter of oak trees of different ages :

Age in years - -	10	20	30	50	70	100	150	200
Diameter in inches	5	10	14	23	32	41	54	64

3. The average daily receipts of a certain grocer :

Day - - -	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.
Receipts in £ -	13	11	14	6	12	18

4. The average weight of boys of different ages :

Age in years -	11	12	13	14	15	16	17	18
Weight in lb. -	79	85	92	102	114	129	142	146

Use of Squared Paper

Both time and trouble are saved by drawing graphs on squared paper.

Example 2. Full marks each week in a class is 100; the following table shows the marks obtained by a boy in successive weeks of a term:

Number of week	1	2	3	4	5	6	7	8	9	10	11
Marks obtained	56	62	81	54	60	52	78	83	65	70	72

Represent this table by a graph.

First draw a line across the paper and mark points at convenient intervals along it to represent the 1st week, 2nd week, 3rd week, etc.

Next draw a line up the page and graduate this to show the marks obtained. *Since no mark is less than* 50 *or greater than* 90, *it is unnecessary to show any graduation outside these limits.*

These two lines are called the **axes of reference** and the graduations show the chosen **scales.**

Lastly draw lines up the page whose lengths represent the marks obtained at the times indicated.

Weekly Marks.

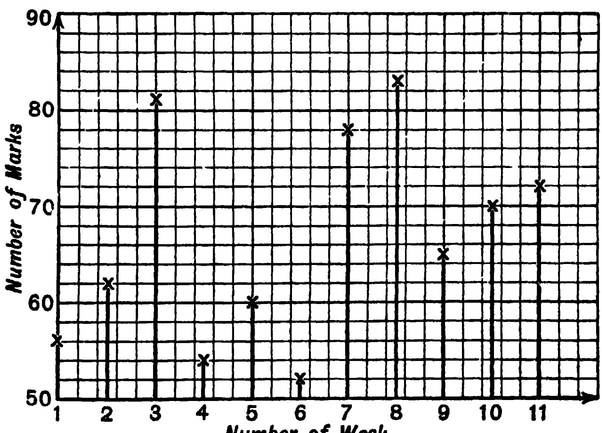


FIG. 81.

***Oral Work, using Fig.* 81.**

(i) In how many weeks did he obtain more than 70 marks?

(ii) In how many weeks did he obtain less than 60 marks ?

(iii) In which week did he obtain (*a*) most marks, (*b*) fewest marks ?

(iv) Would an upright midway between the 3rd and 4th uprights have any meaning ?

Axes and Scales

Axes. A graph records how one quantity varies in size when another quantity, on which it depends, varies. We select values of the latter quantity and then find by observation, or measurement, or calculation, the corresponding values of the former quantity. *The axis for the quantity, whose values we select, should always be drawn* **across the page,** *and the axis for the quantity, whose values are observed or calculated, should be drawn* **up the page.** Thus, in the example on p. 65, we *select* special times and then *observe* the height in the gauge at these times. The time-axis is therefore, in this case, drawn across the page.

The quantity whose values we select is called the **independent** variable, the quantity whose values we observe or calculate is called the **dependent** variable.

Scales. *Great care is needed in the choice of scales* : a bad scale may make the graph worthless.

First, see what range of values is to be represented ; then see what length of axis is available for graduation. Thus in Example 2, p. 67, the marks obtained are never less than 50 or more than 90. The lowest graduation on the upright axis is therefore taken as 50 and the highest as 90 ; also $90-50=40$; $\therefore$ for an upright axis, 2 inches long, we take 1 inch to represent 20 marks. If the axis had been graduated from 0 to 100, it would have been necessary to take 1 inch to represent 50 marks ; this scale would be inconveniently small.

Choose a scale which makes plotting and reading easy. 1 inch may be taken to represent 1, 10, 100, etc., or 0·1, 0·01, etc., or 2, 20, etc., or 5, 50, etc., or 0·2, 0·5, etc. *Occasionally,* 1 inch may be taken to represent 4, 40, etc., but this is not so easy a scale to work with ; 1 inch should *not* be taken to represent 3 or 7, etc.

The object in representing facts graphically is to convey information *rapidly*. *To make a graph easy to understand*, the following instructions must always be carried out.

(i) Write above the graph a title or a brief explanatory heading.

(ii) The quantity whose values are selected must be measured along the axis across the page; the quantity whose values are observed, or calculated, along the axis up the page.

(iii) Write along each axis what that axis represents.

(iv) Choose as large a scale as the paper will allow, but it must be a scale which makes plotting and reading easy.

(v) Graduate each axis so as to show clearly the scale for that axis.

EXERCISE V. c

State which quantity should be measured along the axis drawn *across* the page for the following graphs:

1. Postage on parcels of various weights.

2. Number of passengers on the Underground at different times of day.

3. A boy's age and height.

4. The H.P. of a motor-car and the tax on it.

5. A travel graph: distance from home and time of day.

6. A man's age and his expectation of life.

7. Temperature of a patient during the day.

8. Record times for races of various lengths.

9. Rainfall and time of year.

10. Age and cost of immediate annuity.

11. Depth of sea at various distances westwards from Land's End.

12. Stretch of a spiral spring under various loads.

13. A mark reducer: original marks and scaled marks.

14. A graph to convert degrees Fahrenheit to Centigrade.

15. A graph to convert miles to kilometres.

What scales would you choose and what would be the smallest and largest graduations, to represent the following ranges of values, for the given lengths of axes?

16. Length 10 inches ; from 7 to 53.

17. Length 10 inches ; from 135 to 1050.

18. Length 10 inches ; from 5·6 to 23·8.

19. Length 6 inches ; from 45 to 100.

20. Length 8 inches ; from 100 to 250.

21. Length 10 inches ; from 0 to 1.

22. Length 5 inches ; from 65 to 295.

23. Length 6 inches ; from 2·75 to 3·25.

24. Length 7 inches ; from 0·28 to 0·54.

25. Length 6 inches ; from 500,000 to 800,000.

Represent on squared paper the following statistics, as in Fig. 81. State in each case (i) whether any meaning can be attached to intermediate upright lines, (ii) whether interpolation is possible with fair accuracy without further data.

26. The number of motor-cars sold by a firm in successive quarters is as follows :

	1928				1929			
Period - -	I	II	III	IV	I	II	III	IV
Number of Cars - -	420	560	620	510	440	630	740	580

What is (i) the best time of year, (ii) the worst time of year for selling cars ?

27. The annual premium for a Life Assurance of £1000 varies with the age of the insurer at the time of the first payment, as follows :

Age - -	25	30	35	40	45
Premium - -	£14. 5s.	£16. 10s.	£19. 5s.	£23	£27. 15s.

Estimate the premiums for starting at the ages of (i) 32, (ii) 38.

28. The number of fatal street motor accidents in England and Wales is shown in the following table :

Year - -	1920	1921	1922	1923	1924	1925	1926	1927
Fatal Accidents -	1812	1825	1958	2205	2750	3032	3662	3947

29. The following table shows the length of the longest day in different latitudes :

Latitude in degrees -	15	25	35	45
Length of day in hours -	12·9	13·6	14·4	15·4
Latitude in degrees -	50	55	60	65
Length of day in hours -	16·1	17·1	18·5	21·2

Estimate the length of the longest day in latitudes 30°, 52°.

In representing a set of statistics by a graph, it is not necessary to draw in the whole of the uprights ; the usual custom is to mark *only the top point of the upright*, and to leave the rest to the imagination.

Example 3. The graph in Fig. 82 shows the number of passengers per hour throughout the day on the London Underground.

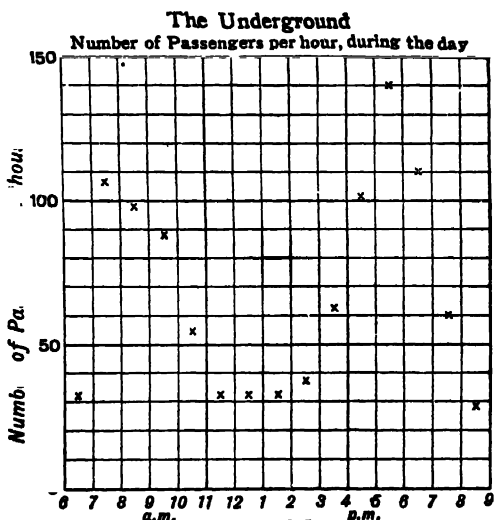


FIG. 82.

Oral work on Fig. 82.

(i) The " workers " are travelling between 7 a.m. and 8 a.m. ; about how many ?

(ii) The business men between 8 a.m. and 10 a.m.; about how many? And late-comers between 10 a.m. and 11 a.m.; about how many?

(iii) When is it most comfortable to travel?

(iv) When are the "rush" hours?

(v) Who are travelling between 5 p.m. and 6 p.m.?

(vi) Doors open at 5 a.m. and close at 1.30 a.m. How could this be shown on the graph?

(vii) This graph does not apply to Saturday or Sunday. What would be the chief differences (*a*) for Saturday, (*b*) for Sunday?

EXERCISE V. d

Represent on squared paper, as in Fig. 82, the following statistics:

State in each case (i) whether any meaning can be attached to intermediate points, (ii) whether any intermediate points can be inserted with fair accuracy without further data.

1. The income-tax paid by a man in various years.

Year - -	1921	1922	1923	1924	1925	1926	1927	1928
Tax in £ -	212	225	247	203	235	192	208	195

2. In 1926, the Civil Service annual salaries were supplemented by a bonus on the following scale:

Bonus in £ - -	99	123	147	176	195	205
Salary in £ - -	200	300	400	600	800	1000

Is the bonus or the salary the *independent* variable?

3. The number of pupils who left and entered a school during successive years was as follows:

Year - - -	1922	1923	1924	1925	1926	1927	1928
Number leaving -	164	147	173	169	152	144	177
Number entering -	153	165	171	156	174	168	180

Draw both graphs on the same figure; mark points on the graph of "numbers leaving" by crosses and points on the graph of "numbers entering" by dots enclosed in circles.

4. Using the data of No. 3, and the fact that there were 810 pupils in the school on Dec. 31, 1921, show graphically the total numbers of the school at the end of each of the years 1922 to 1928.

5. The population of Australia is recorded as follows :

Year	1871	1831	1891	1901	1911	1921
Population in millions	1·66	2·25	3·17	3·77	4·45	5·44

Estimate roughly the population in 1896 and 1907.

6. The following table compares the death-rate of first-class cricketers with the general (male) death-rate, by giving the number per 1000 who die between various ages :

Age	25-35	30-40	35-45	40-50	45-55	50-60
Cricketers	35	46	61	83	117	122
General	47	60	78	102	138	190

Draw both graphs on the same figure, see No. 3.

Locus Graphs

Fig. 83 shows the temperature, at stated times, of a boy with a feverish cold. If his temperature had been taken more frequently,

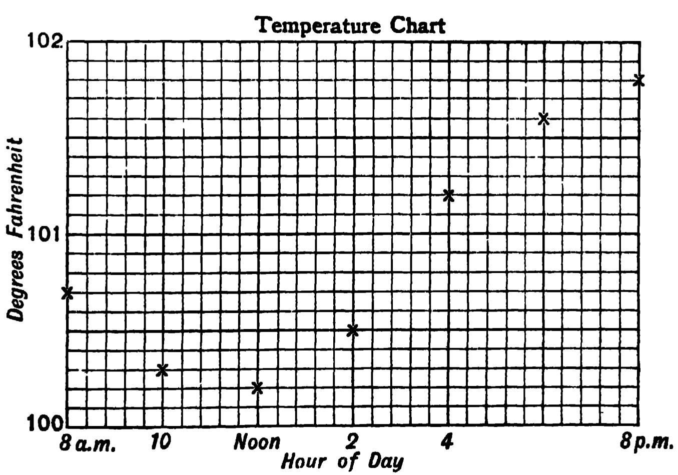


FIG. 83.

there would be more points marked on the graph ; but there are sufficient to give a good idea of how his temperature is changing.

Draw *in pencil* a smooth curve through the marked points in Fig. 83. This curve probably represents with fair accuracy the locus of the top points of the uprights which correspond to the temperatures at intermediate times. We therefore call it a **locus-graph.** It shows at a glance the boy's temperature (approximately) at any time between 8 a.m. and 8 p.m.

Oral work on Fig. 83.

(i) What is approximately his temperature at 9 a.m., 5 p.m. ?

(ii) At what time approximately is his temperature 101°, 101·7° ?

(iii) At what times approximately is his temperature 100·4° ?

The top points of successive uprights are often joined by straight lines in order to guide the eye rapidly from one point to the next; in such cases the intermediate points on the lines do not represent intermediate observations.

Similarly, we may join the top points of the uprights in Fig. 81, p. 67, by straight lines in order to show clearly the ups and downs in successive weeks, although here the intermediate points on the lines have practically no meaning.

Example 4. The height of the barometer in inches is recorded at hourly intervals on a certain day as follows:

Time - -	9 a.m.	10 a.m.	11 a.m.	12	1 p.m.	2 p.m.	3 p.m.
Height in inches -	29·55	29·70	29·77	29·70	29·90	29·72	29·15

We select the times at which the height is observed, the time-axis is therefore taken across the page; unit, 1 inch represents 2 hours, the first graduation is 9 a.m.

All readings lie between 29 in. and 30 in.; ∴ the lowest graduation on the axis up the page may be taken as 29 in.; scale, 1 inch along axis represents 0·5 in., height of barometer.

The given observations are represented by the points, marked by crosses, in Fig. 84.

If an automatic recording machine had been employed, the pointer of the machine would have marked not only these isolated

points but also a continuous curve passing through them, thus forming a locus-graph ; this is shown in Fig. 84.

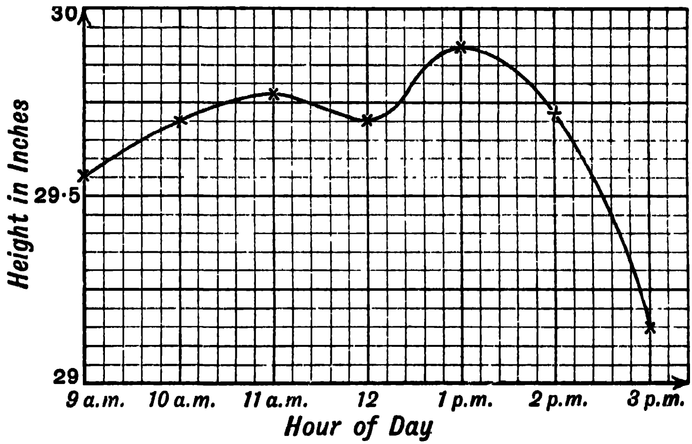


FIG. 84.

Oral work on Fig. 84.

(i) What is the height of the barometer at 9.36 a.m., 12.24 p.m., 1.24 p.m., 2.48 p.m. ?

(ii) At what times is the height of the barometer 29·65 in., 29·80 in., 29·45 in. ?

(iii) Between what times was the barometer rising ?

(iv) Between what times was the barometer above 29·65 in. ?

(v) How much did the barometer fall between 1.30 p.m. and 2.30 p.m. ?

(vi) How much did the barometer rise between 9.30 a.m. and 10.30 a.m. ?

(vii) What inferences can you draw from noticing that a special part of the graph *slopes downwards* and that one part slopes downwards *more steeply* than another part ?

EXERCISE V. e

1. The following table gives the distance d yards in which a train running at V miles per hour can be stopped :

V	30	40	45	50	60
d	100	176	223	276	400

Find from a graph (i) how much further a train runs after the brakes are put hard on when the speed is 35 m.p.h., 55 m.p.h.; (ii) how fast a train is travelling if it can be stopped in 200 yards.

Compare the *extra* distances that must be allowed, for an increase of velocity of 1 mile an hour, for two trains travelling at 35 m.p.h. and 55 m.p.h. respectively.

2. If £1 is allowed to accumulate at 4 per cent. per annum compound interest, the amount is as follows:

Number of years -	0	5	10	20	30	35
Amount in £ - -	1	1·22	1·48	2·19	3·24	3·95

Find from a graph the amount after (i) 15 years, (ii) 25 years, (iii) 33 years.

After what time will £1 amount to £3. 10s.?

Draw on the same figure a graph showing the amount of £1 if allowed to accumulate at 4 per cent. per annum, *simple* interest, for the same period.

3. Expectation of life of an Englishman at different ages.

Age in years	30	40	50	60	70	80	90
Expectation in years -	33·2	26·5	19·9	13·6	8·6	5·2	2·8

Find from a graph (i) how much longer an Englishman may expect to live at the age of 34, 53, 66; (ii) at what age the expectation of life is 22, 16, 11.

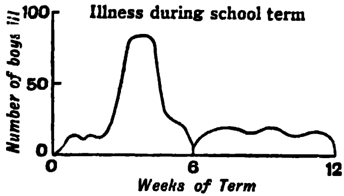
4. The time of a complete oscillation for pendulums of different lengths, in London, is as follows:

Length in ft. - -	1	2	3	4	5	6
Time in sec. - -	1·11	1·57	1·92	2·21	2·48	2·71

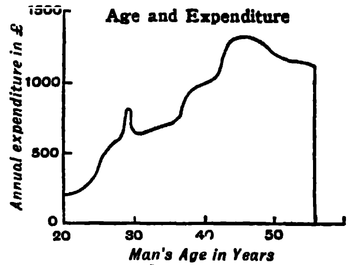
Find from a graph (i) the time for lengths 2 ft. 6 in., 4 ft. 9 in.; (ii) the length to give a time, 2 sec.

The pendulum of a clock should make complete oscillations every 2 seconds if the clock is keeping time. How should the length be corrected if a complete oscillation takes 2·1 seconds? What alteration in length is required to reduce the time of a complete oscillation from 1·3 sec. to 1·2 sec.? Is it the same as before?

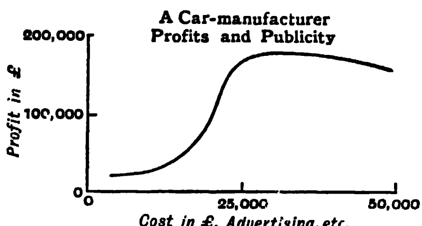
5–8. Describe in general terms the following rough graphs, Nos. 5-8, and explain any peculiar features.



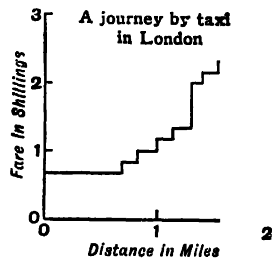
5. FIG. 85.



6. FIG. 86.



7. FIG. 87.



8. FIG. 88.

9. Fig. 89 shows some points on the travel-graph of a steamer. How far did the steamer travel in (i) 2 hours, (ii) 4 hours, (iii) 6 hours ? Use your ruler to show that all the marked points lie

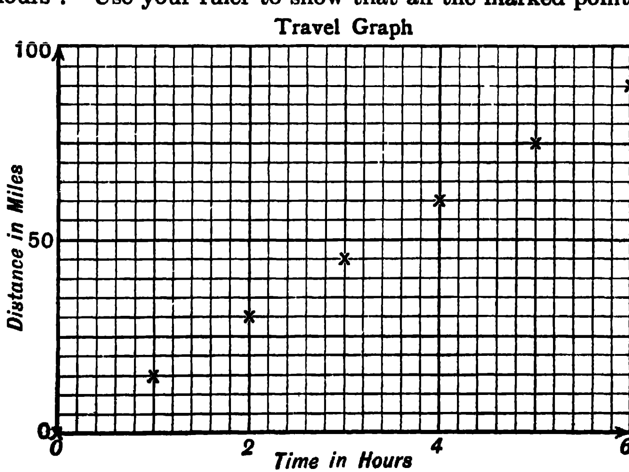


FIG. 89.

on a *straight* line. What does this mean ? What is the speed of the steamer ?

Draw, *in pencil*, travel-graphs for speeds of 50 m.p.h., 25 m.p.h., 5 m.p.h.

10. Interpret the travel-graph in Fig. 90, stating the different speeds in miles per hour. What is the average speed for the whole journey

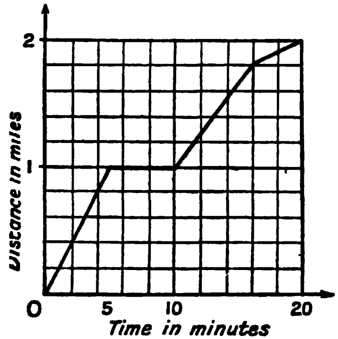


FIG. 90.

How can you tell without any calculation which part of the graph corresponds to the greatest speed ?

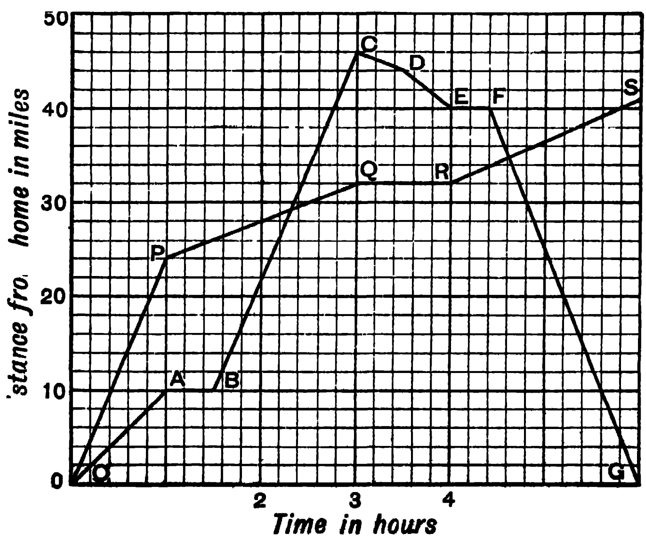


FIG. 91.

11. Fig. 91 shows two travel-graphs. The graph OCG corresponds to a man who successively cycles, motors, walks, takes a bus and a train; the graph OQS to a man who motors and then walks. Both men start from the same house at 10 a.m.

(i) Take the graph OCG and describe the journey in detail, giving the various speeds.

(*a*) Can you tell at a glance when he is moving fastest and when he is walking?

(*b*) How far does he go altogether? When does he start to come home?

(ii) Repeat (i) for the graph OQS.

(iii) If the two men pass one another, when and where does it happen?

12. A motorist leaves home at 9 a.m.; the mileage recorded by his cyclometer, originally set at zero, is as follows:

Time -	9.10	9.20	9.25	9.30	9.40	9.50	10.0	10.10	10.20
Mileage	2	5	9	12	16	19	24	31	37

Find from a graph (i) the readings at 9.36, 9.54, 10.15 a.m., (ii) the distance travelled between 9.35 a.m. and 10.5 a.m., (iii) the time when the distance travelled is 14 mi., 21 mi., 33 mi., (iv) what is his approximate speed at 9.40 a.m., 10.10 a.m.?

13. A spiral spring is suspended from one end and its length is measured when different weights are attached to the other end.

Weight in gm. -	10	15	30	50	75
Length in cm. -	22	24	30	38	48

Represent graphically the relation between the length and the load.

What is the length if the load is 20 gm., 40 gm., 65 gm.?
What is the load if the length is 23 cm., 28 cm., 42 cm.?
What is its natural length, *i.e.* the length when there is no load?
Is the graph a straight line? If so, what does this mean?
How can you interpret the slope of the line?

14. The British amateur running records are as follows:

Distance in yd. -	150	200	440	600	880	1000
Time in sec. -	14·6	19·4	47·0	70·8	112·2	132·4

Represent this table by a graph. [P.T.O.]

What would be the probable record for 500 yd., 750 yd. ?

The American record for 300 metres (1 m. $=1\cdot09$ yd.) is $33\cdot2$ sec. ; how does this compare with British records ?

Should this graph be a straight line ?

Draw *rough graphs* to illustrate the following, Nos. 15-18 :

15. A travel-graph : a boy walks for 5 minutes, runs for 2 minutes, stands still for 3 minutes, and then returns to his starting-point in a car.

16. The inland postage for letters of various weights : charge, up to 2 oz., $1\frac{1}{2}$d. ; for each additional 2 oz. (or part of it), $\frac{1}{2}$d. more.

17. The change of depth of water in a tank, into which the rain-water from the roof of a house runs : fine early, a heavy shower at breakfast, fine between breakfast and lunch, a drizzle after lunch and a steady downpour after tea which fills the tank by 7 p.m.

18. The time a person takes to run a hundred yards at different ages.

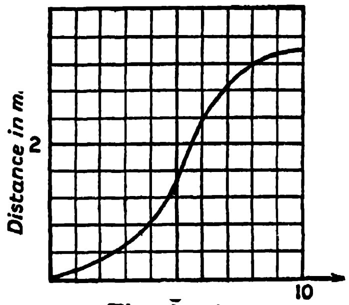


FIG. 92.

19. Draw on an enlarged scale the travel-graph in Fig. 92. State the distance travelled in the first 4 minutes, in the first 6 minutes, in the first 8 minutes. Find in miles per hour (i) the average speed for the first 4 minutes, (ii) the approximate speed after 3 minutes, (iii) the approximate speed after 8 minutes.

After what time is the speed greatest ?

20. The amount of water in a tank, t seconds after the escape pipe is opened, is n gallons where t, n are related as follows :

t	0	5	10	15	20	30	40	50	60	70
n	64	$56\cdot2$	49	$42\cdot3$	36	25	16	9	4	1

Find from a graph (i) how many gallons run out in the first 25 seconds, and in the first 55 seconds ; (ii) the approximate rate at which the water is running out, in gallons per second, after 15 seconds and after 30 seconds.

CHAPTER VI

BRACKETS

Brackets show the order in which operations must be performed If the brackets are absent, different operations may be necessary.

Removal of Brackets. Addition and Subtraction.

$(12-6)-4$ means "From 12 subtract 6, then subtract 4 from the result." This is usually written $12-6-4$, because the omission of the brackets does not cause any error. Thus, instead of $(x-y)-z$, we write $x-y-z$, etc.

The expressions $\mathbf{x-(y+z)}$ and $\mathbf{x-(y-z)}$.

If I have 12 pence and spend $(6+4)$ pence on a ticket, I have $[12-(6+4)]$ pence left. The result would have been the same if I had spent 6 pence on one ticket and 4 pence on another ticket ; the amount then left would be written $[12-6-4]$ pence ;

$$\therefore\ 12-(6+4)=12-6-4.$$

In general, $$\mathbf{x-(y+z)=x-y-z.}$$

If I have 12 pence and buy a ticket for $(6-4)$ pence, I have $[12-(6-4)]$ pence left. The result would have been the same if I handed over 6 pence and received back 4 pence change ; the amount then left would be written $[12-6+4]$ pence ;

$$\therefore\ 12-(6-4)=12-6+4.$$

In general, $$\mathbf{x-(y-z)=x-y+z.}$$

In the same way, we can show that

$$\mathbf{x+(y+z)=x+y+z} \quad \text{and} \quad \mathbf{x+(y-z)=x+y-z.}$$

Thus $12+(6+4)=12+10=22$ and $12+6+4=18+4=22$.

Also $12+(6-4)=12+\ 2=14$ and $12+6-4=18-4=14$.

The rules for the removal of brackets are therefore as follows :

(i) If a bracket has a + sign in front of it, any sign connecting terms in the bracket remains the same when the bracket is removed.

(ii) If a bracket has a − sign in front of it, any sign connecting terms in the bracket is changed when the bracket is removed.

Thus $a+(b-c+d)=a+b-c+d$ and $a-(b-c+d)=a-b+c-d$.

EXERCISE VI. a (Oral)

What are the values of the following ?

1. $(5+6)-2$; $5+(6-2)$.
2. $(10-2)-5$; $10-(2+5)$.
3. $(6+2)+3$; $6+(2+3)$.
4. $(8-5)+2$; $8-(5-2)$.
5. $9-(2+3)$; $9-2-3$.
6. $10+(6-2)$; $10+6-2$.
7. $3+(4+5)$; $3+4+5$.
8. $7-(5-1)$; $7-5+1$.

State in words the meanings of :

9. $(a+b)-c$.
10. $(a-b)-c$.
11. $(a-b)+c$.
12. $a+(b-c)$.
13. $a-(b+c)$.
14. $a-(b-c)$.

Remove brackets from the following :

15. $r+(s-t)$.
16. $r-(s+t)$.
17. $(r-s)-t$.
18. $(a-b)+(c-d-e)$.
19. $(a-b)-(c+d-e)$.

20. A man has £x in the Bank ; he draws out £y and, later, hands back £z. How much has he left in the Bank ? How is the result expressed if he merely draws out £$(y-z)$?

21. From a tank containing N gallons of water, I first draw off p gallons and then q gallons. How much is left ? How is the result expressed if $(p+q)$ gallons are drawn off, all at one time ?

Copy and complete the following :

22. $l+m-n=l+(\quad)$.
23. $p-q-r=p-(\quad)$.
24. $r-s+t=r-(\quad)$.
25. $b+c+d=b+(\quad)$.

Simplify the following :

26. $a+(a+2b)$.
27. $3a-(a+b)$.
28. $2b+(c-b)$.
29. $p+(q-p)$.
30. $1+t-(1-t)$.
31. $r-(s+2t)$.
32. $(\mathrm{R}+3)-(\mathrm{R}-1)$.
33. $x+2y-(x+y)$.
34. $2a+(3-a)$.
35. $1-(b-1)$.
36. $2c-(c-d)$.
37. $3x+(2x-2y)$.
38. $(r+s)-(r-s)$.
39. $p-(q-3p)$.
40. $4a+(2b-a)$.
41. $(p+q+r)+(p+q-r)$.
42. $(x+y+z)-(x-y-z)$.
43. $(a+b-c)-(a-b+c)$.
44. $(r-s+t)+(t+s-r)$.
45. $(x-y)+(y-z)-(x-z)$.
46. $a+2b+3c-(a-2b-3c)$.
47. $2p-q-(2q-r)-(r-p)$.
48. $3x-3-(x+1)+(1-x)$.
49. $f+2g-(f-g-h)$.
50. $1-(1-x)-(2-x)$.

Solve the following equations :

51. $x-(7-x)=5$.
52. $(2t-1)-(t+1)=0$.
53. $3r=7-(2+r)$.
54. $4-(3n+1)=3$.
55. $(3p-2)+(5p-3)=1$.
56. $1-(2y-4)+(5-y)=0$.

57. From $a + 2b - c$ take $a - b + 2c$.

58. Take $p - 2q$ from the sum of $3q - p - r$ and $r + 2p - q$.

59. What must be added to $3r - 2s + t$ to make $r + s - 2t$?

60. By how much does $2m - n - s$ exceed $n - m + 2s$?

[*Note. For additional drill-examples, see Exercise E.P.* 8, p. 139.]

Removal of Brackets. Multiplication and Division

Example 1. A room is h ft. high, l ft. long, b ft. broad. Find the total area of the four walls.

Fig. 93, which represents the four walls, folded out flat, shows that the total area of the four walls $= h(2l + 2b)$ sq. ft.

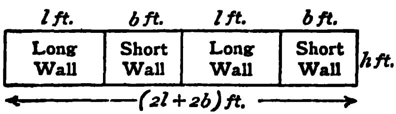


FIG. 93.

But, by taking each wall in turn, we see that the area is

$$(hl + hb + hl + hb) \text{ sq. ft.} = (2hl + 2hb) \text{ sq. ft.}$$

$\therefore$ the area is represented by either of the expressions

$$h(2l + 2b) \text{ sq. ft. or } (2hl + 2hb) \text{ sq. ft.}$$

Again, we see from the figure that half the total area is $h(l + b)$ sq. ft. $\therefore$ the total area $= 2h(l + b)$ sq. ft.

$$\because \quad \mathbf{2h(l + b) = h(2l + 2b) = 2hl + 2hb.}$$

This example shows that when **an expression in a bracket** *is multiplied by a number, each term in the bracket must be multiplied by that number when the bracket is removed.*

It is important to realise that the result of multiplying $h(l + b)$ by 2 may be written,

either as $2h(l + b)$ *or* as $h(2l + 2b)$.

But the expression $2h(2l + 2b)$ is *four* times $h(l + b)$, just as in arithmetic 30×40 is 100 times 3×4.

Example 2. Multiply (i) $2a(3a - 4b)$ by 10; (ii) $\frac{2}{3}(x + 5)$ by 6

(i) The product may be written

either $20a(3a - 4b)$ *or* $2a(30a - 40b)$.

$$20a(3a - 4b) = 20a \times 3a - 20a \times 4b = \mathbf{60a^2 - 80ab.}$$

$$2a(30a - 40b) = 2a \times 30a - 2a \times 40b = \mathbf{60a^2 - 80ab.}$$

(ii) The product equals $\dfrac{6 \times 2(x+5)}{3} = \dfrac{12(x+5)}{3}$

$$= 4(x+5)$$

$$= 4x + 20.$$

Note. $\dfrac{12(6x+30)}{3}$ equals $\dfrac{2}{3}(x+5) \times 36$.

Example 3. Simplify $2(4r-3s)-3(2r+2s)$.

$$2(4r-3s)-3(2r+2s) = (8r-6s)-(6r+6s)$$
$$= 8r-6s-6r-6s = 2r-12s.$$

Note. Mistakes are less likely to occur if the work is done in two steps as above: multiply first and afterwards remove the bracket. *Do not try to do both operations in a single line.*

Example 4. Simplify $\dfrac{2x-1}{4} - \dfrac{3x+1}{7}$.

$$\frac{(2x-1)}{4} - \frac{(3x+1)}{7} = \frac{7(2x-1)}{4 \times 7} - \frac{4(3x+1)}{4 \times 7}$$
$$= \frac{7(2x-1)-4(3x+1)}{28} = \frac{(14x-7)-(12x+4)}{28}$$
$$= \frac{14x-7-12x-4}{28} = \frac{2x-11}{28}.$$

Note. When the process is understood, the working may be shortened, as follows:

$$\frac{2x-1}{4} - \frac{3x+1}{7} = \frac{(14x-7)-(12x+4)}{28} = \frac{14x-7-12x-4}{28}$$
$$= \frac{2x-11}{28}.$$

Example 5. Simplify $\dfrac{a+b}{4a} - \dfrac{a-b}{6b}$.

$$\frac{a+b}{4a} - \frac{a-b}{6b} = \frac{3b(a+b)-2a(a-b)}{12ab}$$
$$= \frac{(3ab+3b^2)-(2a^2-2ab)}{12ab}$$
$$= \frac{3ab+3b^2-2a^2+2ab}{12ab} = \frac{3b^2+5ab-2a^2}{12ab}$$

Example 6. Solve the equation $x - \frac{2x-1}{5} = \frac{x+1}{3}$.

Multiply each side by 15.

$$15x - \frac{15(2x-1)}{5} = \frac{15(x+1)}{3}.$$

$$\therefore\ 15x - 3(2x-1) = 5(x+1).$$

$$\therefore\ 15x - (6x-3) = 5x+5.$$

$$\therefore\ 15x - 6x + 3 = 5x + 5.$$

$$\therefore\ 15x - 6x - 5x = 5 - 3.$$

$$\therefore\ 4x = 2\ ;\quad \therefore\ x = \frac{1}{2}.$$

EXERCISE VI. b

Remove the brackets in the following expressions :

1. $2(3a-4)$.
2. $b(x-y)$.
3. $(c-d)k$.
4. $\frac{1}{2}(4n-2)$.
5. $a(b+2c)$.
6. $(3r-s)r$.
7. $2p(p+q)$.
8. $(x-y)2x$.
9. $2ab(a+b)$.
10. $\frac{1}{3}(6x-9y)$.
11. $3c^2(1-c)$.
12. $3r(3-2s)$.
13. $(a^2-ab)\div a$.
14. $\frac{1}{x}(x^2+2xy)$.
15. $\frac{1}{3}(6b+c)$.
16. $\frac{3}{4}(8R-2)$.
17. $\frac{b}{2}(c+6d)$.
18. $\frac{2a}{3}(2a-9b)$.

Simplify the following :

19. $5(t+1)-3(t-1)$.
20. $3(a-b)-2(a+b)$.
21. $4(r-s)+3(r-s)$.
22. $7c-2(c-d)$.
23. $a(a+2b)-b(a-b)$.
24. $(c+2d)d-(d-3c)c$.
25. $2x(4x-5)-5(x-2)$.
26. $3(2r-s-t)-2(s-t)$.
27. $2c(c+3d)-3d(2c+d)$.
28. $3x(3x-2y)+2y(3x-2y)$.
29. $\frac{x+1}{2}-\frac{x-1}{3}$.
30. $\frac{2a-1}{5}+\frac{1-2a}{7}$.
31. $\frac{1}{4}(R+r)-\frac{1}{6}(R-r)$.
32. $\frac{2}{3}(p+2q)-\frac{1}{2}(p+q)$.
33. $\frac{c}{2d}-\frac{c-d}{3d}$.
34. $\frac{s-2}{4t}+\frac{s-1}{6t}$.
35. $a-\frac{a-b}{2}$.
36. $\frac{3}{4}(d-e)+e$.
37. $1-\frac{5p-3q}{5p}$.
38. $\frac{2y+z}{z}-\frac{y-z}{y}$.

39. $\dfrac{a+x}{2ax}-\dfrac{a-x}{3ax}$.

40. $\dfrac{m}{n}-\dfrac{2n-m}{m}$.

41. $\dfrac{a}{2b}+\dfrac{b}{2a}-\dfrac{a^2+b^2}{6ab}$.

42. $1-\dfrac{c+d}{2c}-\dfrac{c-d}{3c}$.

Solve the following equations :

43. $5(x+1)-3(x-1)=14$.

44. $3-5(y-2)=3(2y-3)$.

45. $7(8-n)-5(11-n)+5=0$.

46. $5(x-2)+2(x-7)=3(x+1)$.

47. $t-\dfrac{2t-3}{7}=4$.

48. $z-\frac{2}{5}(z+1)=5$.

49. $6-\frac{3}{4}(r-7)=\dfrac{r}{2}$.

50. $\frac{1}{2}(x+1)-\frac{2}{3}(x-1)=0$.

51. $\dfrac{p+1}{2}+\dfrac{p-1}{3}=1$.

52. $\frac{2}{3}(y-1)-\frac{1}{4}(y+5)=6$.

53. $\dfrac{3n+1}{5}=\dfrac{5(1+n)}{3}-10$.

54. $\dfrac{2x+1}{5}-\dfrac{x-1}{3}=1$.

55. (i) Evaluate 87×99 by writing it as $87(100-1)$.
(ii) Evaluate in a similar way 999×753.

56. (i) Evaluate $48\times 1\frac{5}{6}$ by writing it as $48(2-\frac{1}{6})$.
(ii) Evaluate in a similar way $42\times 1\frac{6}{7}$.

Copy and complete the following relations :

57. $x+2y-2z=x+2(......)$.

58. $a-3b-3c=a-3(......)$.

59. $2r^2-6rs=2r(......)$.

60. $2p-q+r=2p-(......)$.

61. $x^2-xy-y^2=x^2-y(......)$.

62. $l-3m+6n=l-3(......)$.

63. $2a-6b+3c=2(......)+3c$.

64. $1-x-x^2=1-x(......)$.

[*Note. For additional drill-examples, see Exercise E.P.* 9, *p.* 140.]

Binomial Products

The systematic treatment of products, quotients and factors is given in Ch. XI, but it may be useful to show here how the methods, just practised, can be applied to binomial products.

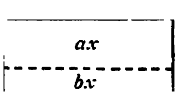


FIG. 94.

Draw a rectangle $(a+b)$ in. high and x in. wide ; its area equals $(a+b)\times x$ sq. in., and is the sum of the areas of the two compartments in the diagram, ax sq. in. and bx sq. in.

Hence $$(a+b)x=ax+bx.$$

If the rectangle is $(c+d)$ in. wide, we write $(c+d)$ instead of x.

Hence $$(a+b)(c+d)=a(c+d)+b(c+d).$$

But the rectangle whose area is $(a+b)(c+d)$ sq. in. can be divided into four compartments whose areas are ac sq. in., ad sq. in., bc sq. in., bd sq. in.

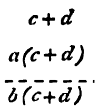


FIG. 95.

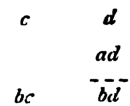


FIG. 96.

Hence $$(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d})=\mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd}.$$

Similarly, $$(a-b)y=ay-by.$$

Now write $(c-d)$ instead of y,

then $$\begin{aligned}(a-b)(c-d)&=a(c-d)-b(c-d)\\&=(ac-ad)-(bc-bd)\\&=\mathbf{ac}-\mathbf{ad}-\mathbf{bc}+\mathbf{bd}.\end{aligned}$$

This result should be illustrated or checked by using Fig. 97.

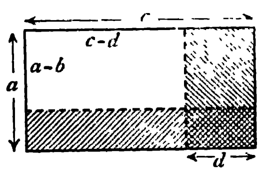


FIG. 97.

Any other binomial product may be worked out in the same way.

Example 7. Expand $(2x-3)(x+5)$.

$$\begin{aligned}(2x-3)(x+5)&=2x(x+5)-3(x+5)\\&=(2x^2+10x)-(3x+15)\\&=2x^2+10x-3x-15=\mathbf{2x^2+7x-15}.\end{aligned}$$

EXERCISE VI. c

Expand the following expressions :

1. $(b+c)(y+z)$.	**2.** $(a-b)(x+y)$.	**3.** $(c+d)(y-z)$.
4. $(a-d)(x-z)$.	**5.** $(a-x)(a+y)$.	**6.** $(c-y)(d-y)$.
7. $(a+b)(a-c)$.	**8.** $(x-p)(x-q)$.	**9.** $(x-a)(x+b)$.
10. $(a+2)(a+3)$.	**11.** $(b+4)(b-1)$.	**12.** $(c-3)(c-5)$.

13. $(x-3)(x+4)$.
14. $(y-1)(y+1)$.
15. $(z+3)(z-3)$.
16. $(p+3)(p+3)$.
17. $(q-5)(q-5)$.
18. $(2+r)(4-r)$.
19. $(a-2b)(a-3b)$.
20. $(c+d)(c-4d)$.
21. $(x-y)(x+3y)$.
22. $(2x+1)(3x-1)$.
23. $(3y+2)(2y+3)$.
24. $(2z-1)(4z-3)$.
25. $(x+y)^2$.
26. $(x-y)^2$.
27. $(x+y)(x-y)$.
28. $(3c-2d)^2$.
29. $(2a+5b)^2$.
30. $(3p+4q)(3p-4q)$.

Systems of Brackets

If one set of brackets is enclosed inside another set of brackets. the meaning of the expression is more easily understood if different shapes of brackets are used. (Cf. p. 14.)

The ordinary forms of brackets employed are as follows :

$$a-(b+c)\ ;\ a-[b-c]\ ;\ a-\{b+c\}\ ;\ a-\overline{b+c}.$$

The use of the line in the last expression is similar to its use in the fraction $\frac{b+c}{2}$, which means $\frac{1}{2}(b+c)$.

Example 8. A rectangle, l in. long, b in. broad, has the same perimeter as a square. By how much does the area of the square exceed that of the rectangle ?

l in.

b in.

FIG. 98.

The perimeter of the rectangle is $2(l+b)$ in.

$\therefore$ the side of the square is $\frac{1}{4}$ of $2(l+b)$ in. $=\frac{1}{2}(l+b)$ in.

$\therefore$ The area of the square is $\{\frac{1}{2}(l+b)\}^2$ sq. in.

$\therefore$ the area of the square exceeds the area of the rectangle by

$$[\{\tfrac{1}{2}(l+b)\}^2-lb] \text{ sq. in.}$$

This expression could also be written, $\left[\left(\frac{l+b}{2}\right)^2-lb\right]$ sq. in.

Example 9. Express in bracket form the result of subtracting $2a-(b-2c)$ from $5a-(b+c)$, and simplify it.

The result of the subtraction may be written

$$\{5a-(b+c)\}-\{2a-(b-2c)\}.$$

This expression $=\{5a-b-c\}-\{2a-b+2c\}$

$=5a-b-c-2a+b-2c=3a-3c.$

When simplifying expressions containing brackets, enclosed in other brackets, **remove the inner brackets first**, and collect like terms (if any) before removing the outer brackets.

EXERCISE VI. d

Simplify the following expressions :

1. $a-(a-\overline{b-c})$.
2. $p+q-\{p-(p-q)\}$.
3. $3[(2a+b)-(a-b)]$.
4. $2\{3(x+1)-2(x-1)\}$.
5. $2p\{p-\overline{p-2q}-q\}$.
6. $5[x-2y]-2[x-\overline{x-y}]$.
7. $10c-2\{c-3(4-c)\}$.
8. $d-3[d-5(d-1)]$.
9. $3\{b-[2-2(b-1)]+b\}$.
10. $2e-[e-2(e-f)-f]$.
11. $2[a(a-b)-b(a+b)]$.
12. $7r-2(r+2s-\overline{s-2r})$.
13. $\{2a(a+b)-\overline{a^2+ac}\}\div a$.
14. $W-W\{3-(2+e)\}$.

15. If a wave h feet high travels across the sea where the depth is d feet, its velocity is $\sqrt{\{32(d+3h)\}}$ feet per second. What is the velocity of a wave 2 ft. high moving across water 12 ft. deep ?

16. If a clothes line, l feet long, is attached to the tops of two poles, d feet apart, the sag at the middle is approximately $\sqrt{\{\frac{3}{8}d(l-d)\}}$ feet. Find the sag if the poles are 24 ft. apart and if the clothes line is 25 ft. long.

17. What sum of money is represented by £$\{\frac{1}{240}[12(20x+y)+z]\}$, if $x=3$, $y=5$, $z=7$? Give the answer as a compound quantity.

18. What weight is ropresented by $\frac{1}{80}\{(20p+q)4+r\}$ tons, if $p=11$, $q=13$, $r=3$? Give the answer as a compound quantity.

19. The total surface area of a closed circular cone of base-radius r in., height h in., is approximately $\frac{22r}{7}\{r+\sqrt{(h^2+r^2)}\}$ sq. in. What is the total surface area of a cone of height 4 in. and base-diameter 6 in. ?

20. Use the formula in No. 19 to find an expression for the total surface area of a cone of height p inches and base-diameter $\frac{3p}{2}$ inches ?

Express Nos. 21-24 in bracket form and then simplify :

21. Subtract $x-(2y+z)$ from $x+(2z-y)$.

22. What must be added to $2f-3[u-v]$ to give $3u-4[v-f]$?

23. What must be subtracted from $5r(r-s)-s^2$ to leave $s^2-3r(s-r)$?

24. By how much does $4l-3\{m-2n\}$ exceed $m-2\{l+3n\}$?

Areas and Volumes

Mensuration formulae are often expressed in bracket-form, not only for the sake of brevity, but also because this is a more suitable form for computation.

We shall assume the following formulae :

1. **The circumference of a circle of radius r in. is $2\pi r$ in.**

2. **The area of a circle of radius r in. is πr^2 sq. in.**

3. **The volume of a cylinder or prism whose cross-section is A sq. in. and whose length is l in. is Al cu. in.**

When numerical values are required, π may be taken as $\frac{22}{7}$ or $3\cdot 14$.

Example 10. Fig. 99 represents a square plate from which four equal semicircles and one complete circle have been cut away, as shown ; the units are inches. Find the area of the upper surface of the plate and express it in the form, N . a^2 sq. in., where N is a number. Evaluate N correct to one place of decimals.

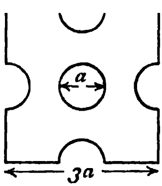


FIG. 99.

Find also the volume of the plate, if its thickness is $\frac{a}{6}$ in.

The area of the surface of the plate, before it is cut, is $3a \times 3a$ sq. in. $= 9a^2$ sq. in.

Four semicircles equal two complete circles ; $\therefore$ in all, 3 complete circles are cut away. The diameter of each circle is a in. ; $\therefore$ the radius is $\frac{a}{2}$ in. ; $\therefore$ the area of each circle $= \pi\left(\frac{a}{2}\right)^2$ sq. in. $= \frac{\pi a^2}{4}$ sq. in.

$$\text{area of surface of plate} = \left(9a^2 - \frac{3\pi a^2}{4}\right) \text{ sq. in.}$$

$$= \left(9 - \frac{3\pi}{4}\right) a^2 \text{ sq. in.}$$

The value of $N = 9 - \frac{3\pi}{4} = 9 - \frac{3 \times 3\cdot 14}{4} = 9 - \frac{9\cdot 42}{4} \simeq 9 - 2\cdot 4 \simeq 6\cdot 6$.

The volume of the plate $= \left(9 - \frac{3\pi}{4}\right)a^2 \times \frac{a}{6}$ cu. in.

$$= \left(\frac{9}{6} - \frac{3\pi}{4 \times 6}\right)a^3 \text{ cu. in.} = \left(\frac{3}{2} - \frac{\pi}{8}\right)a^3 \text{ cu. in.}$$

EXERCISE VI. e

[*Do not substitute for π, unless specially told to do so.*]

1. Fig. 100 represents a rectangular plate with two rectangular holes pierced in it, the units being inches. Express in bracket form the area of the upper surface of the plate. How can the result be written if $q = 5a$?

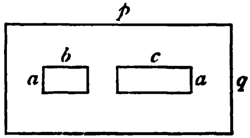


FIG. 100.

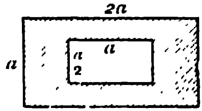


FIG. 101.

2. If the units of the dimensions in Fig. 101 are inches and all corners are right-angled, express the shaded area in the form (i) N . a^2 sq. in., (ii) k . a^2 sq. ft.

3. The shaded area in Fig. 101, is the base of a hollow brick, the units being inches ; and its length is $2a$ inches. Express the volume of the brick in the form k . a^3 cu. in.

4. With the data of No. 2, if the plate weighs W oz., and is made of material weighing w oz. per cu. in., find the thickness of the plate.

5. Find in bracket form the area of Fig. 102 ; state a unit.

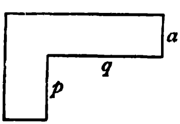


FIG. 102.

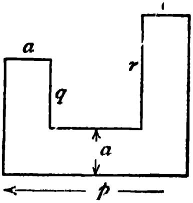


FIG. 103.

6. Find in bracket form the area of Fig. 103 ; state a unit.

7. Fig. 104 represents two congruent right-angled trapeziums

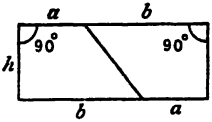


FIG. 104.

fitted together to form a rectangle. What is the area of the whole figure ? What is the area of each trapezium ? State a unit.

8. Fig. 105 is formed of a rectangle with triangles on one pair

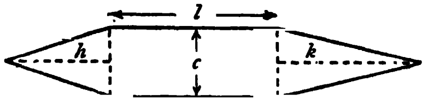


FIG. 105.

of opposite sides as bases. State a unit and find the area of the figure.

9. Fig. 106 represents a lawn surrounded by a path d ft. wide. Express in two different ways the area of the path.

10. What is the total surface area of a cubical block whose edge is (i) l in., (ii) $(l+1)$ in.? Do not remove brackets.

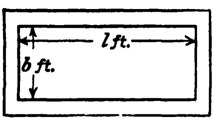


FIG. 106.

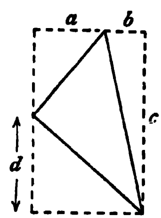


FIG. 107.

11. Find the area of the triangle in Fig. 107, the figure enclosing it being a rectangle. State a unit.

12. A closed box with rectangular faces is l ft. long, l ft. wide, h ft. high. Express the total area of its surface in bracket form.

13. Repeat No. 12, if the length, breadth and height are each increased by 1 foot. Do not remove brackets.

14. In a brick wall, l ft. long, h ft. high, there are two windows of height a ft. and of lengths u ft., v ft. respectively; there are also two windows of height b ft. and of lengths p ft., q ft. respectively. Express by brackets the area of the brickwork.

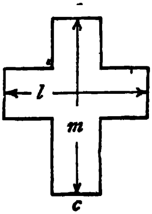


FIG. 108.

15. The cross in Fig. 108 is formed by two overlapping rectangles, the units being inches. Find the area of the upper (visible) surface in bracket form.

16. Fig. 109 represents the quadrant of a circle. Express its perimeter in the form $\mathsf{N} \cdot r$.

17. What is the radius of the circle whose area is equal to that of the quadrant in Fig. 109 ?

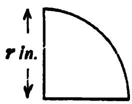


FIG. 109.

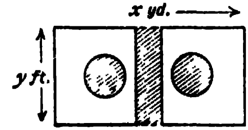


FIG. 110.

18. Fig. 110 represents a lawn divided into two parts by a path 6 ft. wide with two circular flower beds, each of radius $\frac{1}{8}y$ feet. What is the area of the grass in sq. ft. ?

19. With the data of No. 18, find the total length of the grass edging in feet.

20. The minute hand of a clock is $3l$ in. long; how far does its tip move in 40 minutes ?

21. Find the shaded area in Fig. 111 formed by cutting a circle of diameter d in. out of a circle of radius d in.

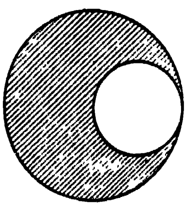


FIG. 111.

22. With the data of No. 21, find the perimeter of the shaded area in Fig. 111.

23. Find the shaded area in Fig. 112 formed by cutting away a quadrant from a square. What is the perimeter of the figuro ?

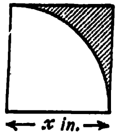


FIG. 112.

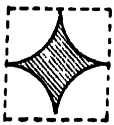


FIG. 113.

24. Find the shaded area in Fig. 113 enclosed by 4 quadrants, each of radius r in. What is the perimeter of the figure ?

25. The total surface area of a closed circular cylinder of radius r in., height h in., is $2\pi r(h+r)$ sq. in. What is this for a cylinder 1 inch high and 1 foot in diameter ? Take $\pi = \frac{22}{7}$.

[*Note. For additional examples, see Appendix, Ex. S.* **6**, p. 288. *For a revision exercise on Ch. III-VI, see Appendix, Ex. R.* **3**, *p.* **261**.]

TEST PAPERS A. 16-25

A. 16

1. (i) Add together $a-b$, $b-c$, $c+a$.

(ii) What is the square of $4x^3y$ and the cube of $2r^2s$?

2. What is the total cost of $2x$ apples at y pence a dozen and $4y$ pears at x pence a dozen ?

3. Solve the equations (i) $\frac{7}{8}(t-11)=3\frac{1}{2}$;

(ii) $5(y-2)+2(y-6)-3(y+5)=5$.

4. Simplify (i) $s-\frac{1}{2}(s+t)$; (ii) $a-3(c-a)$.

5. In Fig. 114, ABC is a straight line ; find the value of x.

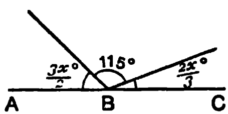


FIG. 114.

A. 17

1. If $s=7$, $t=4$, find the values of

(i) $3s-t^2$; (ii) $2t^2-st$; (iii) t^3-s^2.

2. (i) What is a square root of $36p^6q^8$?

(ii) If $9x$ gallons of beer cost $2y$ shillings, find the cost in pence of half a pint, at the same rate.

3. (i) Solve the equation, $0{\cdot}1(n-3)=0{\cdot}15(n-4)$.

(ii) Show that $y=10$ is a root of the equation

$$y^3-y(y-1)^2=190.$$

4. Simplify (i) $(x^2-3x-2)(3x^2)-(x^3+x-1)(2x)$;

(ii) $1\frac{3}{4}(T+t)-1\frac{1}{4}(T-t)$.

5. A jug and basin cost 11s. ; the jug costs half a crown more than the basin ; find the cost of each.

A. 18

1. (i) Simplify $c-\dfrac{c-1}{3}-\dfrac{c+2}{6}$.

(ii) If $a=3$, $b=2$, verify that $(a-b)^3=a^3-b^3-3ab(a-b)$.

2. A grocer buys 20 dozen eggs at p shillings a dozen ; he sells 16 dozen at $(p+\frac{3}{4})$ shillings per dozen and the rest at $(p+\frac{1}{8})$ shillings per dozen. What is his profit ?

3. (i) Solve the equation, $\dfrac{1}{2z}=\dfrac{2}{z+2}$.

(ii) For what value of c is $x=2$ a root of the equation,

$$3(x-c)-\frac{9c}{x}=\tfrac{1}{2}x\ ?$$

4. Copy and fill in the blanks in

(i) $6p^2-3pq=3p(\qquad)$ (ii) $2a-b+c=2a-(\qquad)$.

5. Fig. 115 shows, in degrees, the angles at which a line is cut

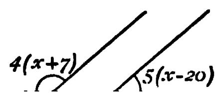


FIG. 115.

by two parallel lines ; find the value of x.

A. 19

1. (i) Simplify $(3r^2)^3-(2r^3)^2$.

(ii) What must be added to $2(2l-m-2n)$ to make

$$5(l+2m-n)\ ?$$

2. The sum of the edges of a cube is l feet. Find (i) the volume of the cube in cu. inches, (ii) the total surface area of the cube in sq. inches.

3. Solve the equations, (i) $3(x-3)-5(12-x)=5$.

(ii) $\dfrac{y-11}{3}-\dfrac{y-3}{4}=\dfrac{1}{2}$.

4. Express $r\{r+\sqrt{(h^2+r^2)}\}$ in terms of p, if $r=3p$ and $h=4p$.

5. (i) Invent a problem about numbers which would lead to the equation

$$n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)=c.$$

Then find n in terms of c.

(ii) Can you find 4 consecutive odd numbers whose sum is 40 ?

A. 20

1. If $r=7$, $s=1$, $t=4$, $v=1$, find the values of
(i) $(r-s)\div(t-v)$; (ii) $r-s\div(t-v)$; (iii) $(r-s)\div t-v$.

2. Tests for the breaking strain of a wire rope gave the following results :

Circumference in inches	1·5	2	2·5	3	3·5	4
Breaking strain in tons	4·0	7·5	12	18	26	37

Find from a graph (i) the breaking strain if the circumference is 2·8 in., 3·7 in. ; (ii) the girth if the breaking strain is 9 tons, 20 tons.

3. Solve the equations, (i) $4x=5x$; (ii) $\frac{7}{t}-3=\frac{5}{t}+2$.

4. Copy and fill in the blanks in
(i) $b-2a=2b-(\quad)$; (ii) $1-x^2+x^6=1-x^2(\quad)$.

5. P $-$ (15 per cent. of 2P) $=21$; find P.

A. 21

1. (i) Find the L.C.M. of $6ab^2c$, $12b^3$, $9a^3bc^2$.
(ii) Multiply $\left(\frac{5}{6n}-\frac{7}{9n}\right)$ by 18.

2. (i) The temperature rises from $t°$ C. to $3t°$ C. ; how many degrees is this ? (ii) The temperature is $(a+b)°$ C. ; if it falls $(a-b)°$, what does it become ?

3. (i) Solve the equation, $x+1-\frac{2x+1}{5}=\frac{x+4}{3}$.
(ii) Find numerical values of x and y such that the expressions $x+2$ and $2y$ and $5-x$ are all equal.

4. Use Fig. 116 to find another way of writing $(t+3)(t+4)$.

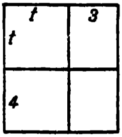


FIG. 116.

5. A bus travels 3 miles an hour faster than a tram. In 80 minutes the tram goes 2 miles further than the bus goes in 50 minutes. Find the speed of the tram in miles per hour.

A. 22

1 If $p=7$, $q=3$, $x=4$, $y=2$, verify that
$$(p-q)(x-y)=(px+qy)-(py+qx).$$

2. The temperature of the water in a boiler is rising at a steady rate. (i) At 6 a.m. it is x°, at 7 a.m. it is y°, what is it at 8 a.m. ? (ii) At 6 a.m. it is p°, at 9 a.m. it is q°, what is it at 7 a.m. ?

3. Solve the equations, (i) $6-\frac{3}{4}(r-6)=\frac{1}{2}(r+1)$;

(ii) $4-\dfrac{3}{s}=\dfrac{18}{s}-3.$

4. (i) What must be added to the sum of $\dfrac{a+b}{4}$ and $\dfrac{a-b}{3}$ to give a ?

(ii) Simplify $(2x)^3 \div (2x)^2$.

5. A boy gets 3 marks for each sum right and loses 2 marks for each sum wrong. He does 24 sums and obtains 37 marks. How many were right ?

A. 23

1. Simplify (i) $5a-\dfrac{a}{5}$; (ii) $3-\dfrac{3b}{b}$; (iii) $\dfrac{3r^2}{s^2}\times\dfrac{s^2}{3r^2}$.

2. (i) A boy was p years old q years ago ; how old will he be in $(p-q)$ years' time ?

(ii) If $\mathrm{R}=3p-q$ and $r=p+3q$, show that $\mathrm{R}^2+r^2=10(p^2+q^2)$ when $p=2$, $q=1$.

3. Solve the equations, (i) $\dfrac{12}{x}=96$;

(ii) $\dfrac{5(1-z)}{3}-\dfrac{3z-1}{5}=\dfrac{1}{6}$.

4. Copy and fill in the blanks in

$$r=\frac{\quad}{3}=\frac{\quad}{s}-=\frac{\quad}{r^2}=r^2+(\qquad).$$

5. I travel 25 miles in $2\frac{1}{2}$ hours, walking part of the way at 3 m.p.h., and motoring the rest at 24 m.p.h. How far do I walk ?

A. 24

1. Simplify (i) $x \div \dfrac{1}{x}$; (ii) $\dfrac{2pq}{3pqr}-\dfrac{q}{6qr}$; (iii) $\dfrac{3s^2}{t^2}\times 3st$.

2. The external dimensions of a closed wooden box are $12a$ in., by $10a$ in., by $8a$ in. ; the wood is a in. thick. Find (i) the area of the outside surface, (ii) the area of the inside surface, (iii) the volume of the space inside the box, (iv) the volume of wood used in making the box.

3. Solve the equations, (i) $\frac{1}{2x}+\frac{1}{4x}=3$;

(ii) $\frac{y-2}{15}+\frac{1}{20}=\frac{y-1}{20}$.

4. By how much does $5x-3(y-2z)$ exceed $6z-3(x+y)$?

5. An article is reduced from P shillings to $\left(\mathrm{P}-\frac{\mathrm{P}}{5}\right)$ shillings in a sale. The sale price is 14s. ; how much is it reduced ?

A. 25

1. Find the value of $3x^2-x+4$ when $x=3, 2, 1, 0$.

2. Experiments with a screw-jack showed that the load to be raised and the necessary effort were related as follows :

Load in lb. wt. -	100	120	160	180	200
Effort in lb. wt. -	12·0	13·8	17·6	19·6	21·5

Find from a graph (i) the necessary effort if the load is 140 lb. wt., 170 lb. wt., 186 lb. wt. ; (ii) the load that can be raised by an effort of 13 lb. wt., 16 lb. wt., 20 lb. wt.

3. Solve the equation, $\frac{1}{3}(x-3)-\frac{1}{4}(5-x)=\frac{1}{6}(4\frac{1}{2}-x)$.

4. (i) What must be added to r tons s cwt. to make up $(r+1)$ tons ?

(ii) Express twice k cwt. as a compound quantity if $10<k<20$

5. There are 2743 candidates at an examination ; some took 4 papers and the rest took 3 papers. There were in all 9115 sets of answers to be corrected. How many took 4 papers ?

CHAPTER VII

DIRECTED NUMBERS

Signless Numbers

In the previous chapters, the signs + and − have been used solely as orders: " add," " subtract." The expression, 5 − 3, means " From 5 subtract 3 " ; the numbers, 5 and 3, are signless. For many kinds of measurement, signless numbers supply all that is needed, *e.g.* the number of days in a week, the number of miles from London to Land's End, etc. In quantities of this kind there is no idea of " up and down " or " backwards and forwards " or " clockwise and anti-clockwise." But when quantities which involve the idea of direction occur, it saves time to give further meanings to the symbols + and −.

Positive and Negative Numbers

Suppose I buy a number of things and sell them again, the results of the transactions may be recorded as follows:

	House	Car	Picture	Horse	Field	Carpet
Gain -	£80			0	£60	
Loss -		£40	£15	0		£12

But it is *shorter* to write :

	House	Car	Picture	Horse	Field	Carpet
Gain -	£(+80)	£(−40)	£(−15)	0	£(+60)	£(−12)

The symbols + and − in this table are not instructions to add or subtract ; they are called the signs of the numbers.

The number (+80) is called a **positive number**, the number (−40) is called a **negative number**. My gain £(+80) is a short way of saying that I am £80 better off, or that my capital *goes up* £80 ; my gain £(−40) is a short way of saying that I am £40 worse off, or that my capital *comes down* £40. Thus this new notation represents in a short form up-and-down movements,

backwards-and-forwards movements, etc., in fact, quantities with which the idea of direction is associated; for this reason positive and negative numbers are called **directed numbers.** The symbol 0 means that there is no change either way, $(+0)$ is the same as (-0) and so each is simply denoted by 0.

EXERCISE VII. a

1. A tank, with its base horizontal, contains water to a depth of 10 in., a number of vertical rods A, B, C, ... are fixed to the base (see Fig. 117).

	A	B	C	D	E	F
Length of rod in inches	18	8	4	12	7	15

Express in short-hand form, the *height* of the upper end of each rod *above* the water level.

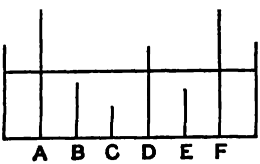


FIG. 117.

2. On a Centigrade thermometer (see Fig. 118), the freezing point of water is indicated by 0° C.; express in short-hand form, the following temperatures:

(i) 4° below freezing point; (ii) 20° above freezing point; (iii) 15·5° above freezing point; (iv) 22·8° below freezing point; (v) $7\frac{1}{2}$ degrees of frost.

What is the meaning of (i) (-5) degrees Centigrade; (ii) $(+18)$ degrees Centigrade; (iii) a rise in temperature of $(-3°)$; (iv) a fall in temperature of $(-4°)$?

FIG. 118.

3. A number of clocks are being regulated; express in short-hand form the following records:

Clock	I	II	III	IV	V	VI	VII
Seconds fast	17		8			3	5
Seconds slow		8		11	14		

4. A gun is being ranged by an aeroplane on a target; the direction is correct; the distance of the fall of the shell from the target is signalled as follows:

Round	I	II	III	IV	V	VI
Distance in yards -	+260	−180	+140	−70	+35	O.K.

What do these signals mean?

5. Explain the following:
Height in feet above sea-level.

Winchester	Dead Sea	Jerusalem	Hill 60, Ypres	Sea of Galilee
(+128)	(−1300)	(+2500)	(+195)	(−680)

6. Express in short-hand form:

To travel from O to - -	A	B	C	D	E	F	G
Proceed miles east - -	24				5	7	
Proceed miles west - -		24	8	10			17

7. Taking the year 1914 A.D. as the beginning of a new era 0 A.B., express the following dates by directed numbers 1913 A.D., 1915 A.D., 1900 A.D., 1926 A.D., the year of your birth.

8. Express in short-hand form a man's banking account:

	Jan. 1	March 1	May 1	July 1	Sept. 1	Nov. 1
Credit -		£70	£30			£12
Overdraft -	£55			£43	£49	

9. What do the following golf handicaps mean:
(i) +4; (ii) −12; (iii) −18; (iv) +1; (v) 0?

10. A stone is thrown vertically upwards with velocity 40 ft. per sec. Explain the meaning of the following table which shows its velocity at half-second intervals:

Time in sec. - -	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
Velocity in ft. per sec.	+40	+24	+8	−8	−24	−40	−56

11. Rewrite the following, using signless numbers:

(i) A temperature of (−8) degrees C.

(ii) The lowest part of the bed of Lake Como is (−600) ft. above sea-level.

(iii) My watch is (− 3) minutes fast, and yours is (− 2) minutes slow.

(iv) My bank balance is £(− 24).

(v) In a 220 yards race, I received a start of (− 10) yards.

(vi) This year, I have gained (− 10) lb. in weight and you have lost (− 15) lb. in weight.

12. Taking Greenwich time as the standard, the following variations occur in local time :

	Berne	Halifax	Petrograd	New York
Hours	+ 1	− 4	+ 2	− 5

What does this mean ?

13. Taking 12 noon as zero hour, express by directed numbers the following times, (i) 3 p.m., (ii) 11 a.m., (iii) 8 a.m., (iv) 4.30 p.m., (v) 10.30 a.m.

14. Write down four consecutive whole numbers of which the greatest is (i) + 10, (ii) + 2, (iii) − 2.

15. A is 2 miles north of B ; how many miles is B north of A ?

The Number-Scale

On the Centigrade scale, the temperature at which water freezes is marked 0 (zero); temperatures above zero are represented by positive numbers, temperatures below zero by negative numbers. This is an example of what is called the *number-scale*, see Fig. 119.

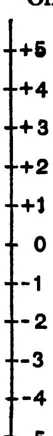


FIG. 119.

Addition and Subtraction

The number-scale can be used to perform addition and subtraction, moving up and down it, as on a ladder.

A is 5 pence in debt, B has 2 pence ; add together what A and B have ; the net result is a debt of 3 pence. This may be written as follows, working in pence :

$$(-5)+(+2)=(-3) \quad \text{or} \quad (+2)+(-5)=(-3).$$

Thus, to add (+ 2) to (− 5), start at (− 5) on the ladder and move up 2 steps ; and to add (− 5) to (+ 2), start at (+ 2) on the ladder and move down 5 steps.

In general, $+(+\text{N})$ means "add $(+\text{N})$"; to do this, move N steps up the scale.

and $+(-\text{N})$ means "add $(-\text{N})$"; to do this, move N steps down the scale.

For subtraction, since

$$(-3)=(-5)+(+2) \text{ and } (-3)=(+2)+(-5),$$

we can say,

$$(-3)-(+2)=(-5) \text{ and } (-3)-(-5)=(+2).$$

Thus, to subtract $(+2)$ from (-3), start at (-3) on the ladder and move down 2 steps;

and to subtract (-5) from (-3), start at (-3) on the ladder and move up 5 steps.

In general, $-(+\text{N})$ means "subtract $(+\text{N})$"; to do this, move N steps down the scale.

and $-(-\text{N})$ means "subtract $(-\text{N})$"; to do this, move N steps up the scale.

The process of moving N steps up the scale may be represented more shortly by writing $+\text{N}$ and that of moving N steps down the scale by $-\text{N}$. We therefore make the following **rule of signs**.

$+(+\mathbf{N})=+\mathbf{N}$; $-(-\mathbf{N})=+\mathbf{N}$; move up N steps.

$+(-\mathbf{N})=-\mathbf{N}$; $-(+\mathbf{N})=-\mathbf{N}$; move down N steps.

EXERCISE VII. b

1. The temperature (Centigrade) is $(+5°)$. What does it become after (i) a rise of $(+7°)$; (ii) a fall of $(+6°)$; (iii) a rise of $(-2°)$; (iv) a fall of $(-3°)$?

What are the values of the following:

(i) $5+(+7)$; (ii) $5-(+6)$; (iii) $5+(-2)$; (iv) $5-(-3)$?

2. Give, in full, expressions for the final temperature, and then simplify them.

(i) First temperature $(-3°)$ C.; a rise of $(+5°)$.
(ii) First temperature $(+5°)$ C.; a fall of $(-2°)$.
(iii) First temperature $(+2°)$ C.; a fall of $(+3°)$.
(iv) First temperature $(-4°)$ C.; a rise of $(-5°)$.

3. A is $(+100)$ feet above sea-level; what is the height of B above sea-level if B is (i) 80 ft. above A, (ii) 60 ft. below A, (iii) 160 ft. below A? What is the value of $(+100)+(-60)$ and of $(+100)+(-160)$?

4. A man starts the year with a deficit of £100 and ends the year with a deficit of £60. What has he gained during the year? What is the value of $(-60)-(-100)$?

5. A man walks up from the basement, 10 ft. below ground-level, to his bedroom, 15 ft. above ground-level. What height has he ascended? What is the value of $(+15)-(-10)$?

6. A freezing mixture is $(-12°)$ C.; it is melted and heated up to $(+4°)$ C. What is the change of temperature?

7. Which is the greater, (-3) or (-6)? Illustrate your answer by referring to (i) the number-scale, (ii) the idea of sea-level.

8. What must be added to

(i) $(+2)$ to give $(+5)$; (ii) $(+2)$ to give (-3);
(iii) (-4) to give (-1); (iv) (-4) to give $(+1)$;
(v) (-5) to give (-9); (vi) (-5) to give 0;
(vii) $(+3)$ to give 0; (viii) 0 to give (-2)?

9. Write down the values of:

(i) $(+8)+(-4)$; (ii) $(-7)+(-3)$; (iii) $(+2)-(+4)$;
(iv) $(-2)-(-3)$; (v) $(-6)-(-5)$; (vi) $(-9)+(+9)$;
(vii) $(-2)-(+3)$; (viii) $0-(+2)$; (ix) $(-2)-(+2)$;
(x) $(-3)-0$; (xi) $(+2)-(-3)$; (xii) $(-6)-(-6)$;
(xiii) $(-5)+(-5)$; (xiv) $(+4)+0$; (xv) $0+(-4)$.

Simplify the following:

10. $(+5t)+(-t)$. 11. $(-3c)+(+c)$. 12. $(-2e)+(-4e)$.

13. $(+2p)+(-8p)$. 14. $(+r)+(-r)$. 15. $(+5s)-(-2s)$.

16. $(-2x)-(+x)$. 17. $(+z)-(+2z)$. 18. $(-4l)-(-7l)$.

19. $0-(+2s)$. 20. $(-3x)-(-3x)$. 21. $0-(-3r)$.

22. $(-b^2)-(+b^2)$. 23. $(+2a^2)-(-2b^2)$. 24. $(-3pq)+(-3pr)$.

25. $a-5a+3a$. 26. $2b-5b+3b$. 27. $3c-5c-4c$.

28. What is (i) the term of degree 2, (ii) the coefficient of x, (iii) the constant term, in the following expressions:

(a) $3x^2-7x-2$; (b) $2x^3-5x^2-x+4$; (c) x^4-x^2+3x-1?

29. What is (i) the term of highest degree, (ii) the coefficient of x^2, (iii) the constant term, in the following expressions:

(a) $10x-x^3-2x^2-3$; (b) $5+2x^4-x^2+12x$?

30. Arrange in descending powers of x,

(a) $6x-2-x^3+5x^2$; (b) $5-2x^4+10x-x^3$.

Multiplication and Division

The temperature of the water in a boiler is being raised at a steady rate of 5° C. per hour throughout the day. If we regard mid-day as zero hour, the temperature at n o'clock is $(+5°)\times n$ above that at mid-day, (zero hour).

Here, n is a directed number: thus at 2 p.m., $n=(+2)$ and at 9 a.m., $n=(-3)$.

At 2 p.m., the temperature is evidently 10° *above* that at mid-day;

$$\therefore\ (+5)\times(+2)=(+10).$$

And at 9 a.m., the temperature is evidently 15° *below* that at mid-day;

$$\therefore\ (+5)\times(-3)=(-15).$$

Next, suppose that the temperature of the water in the boiler is *falling* at a steady rate of 5° C. per hour throughout the day. Then the *rise* per hour is $(-5°)$; $\therefore$ the temperature at n o'clock is $(-5°)\times n$ above that at mid-day, (zero hour).

At 2 p.m., the temperature is evidently 10° *below* that at mid-day;

$$\therefore\ (-5)\times(+2)=(-10).$$

And at 9 a.m., the temperature is evidently 15° *above* that at mid-day;

$$\therefore\ (-5)\times(-3)=(+15).$$

This argument can be applied to any directed numbers; we therefore make the following rule of signs:

$$(+\mathbf{a})\times(+\mathbf{b})=(+\mathbf{ab})=\mathbf{ab};\quad(-\mathbf{a})\times(-\mathbf{b})=(+\mathbf{ab})=\mathbf{ab}.$$
$$(+\mathbf{a})\times(-\mathbf{b})=(-\mathbf{ab})=-\mathbf{ab};\quad(-\mathbf{a})\times(+\mathbf{b})=(-\mathbf{ab})=-\mathbf{ab}.$$

where, for simplicity, we write ab for $+ab$.

Oral Example. The temperature at mid-day is 60° C. What are the temperatures at 8 a.m. and at 5 p.m. if the temperature (i) is rising, (ii) is falling, at a steady rate of 3° C. per hour?

Division

Since $(+5)\times(+2)=(+10)$, $\therefore\ (+10)\div(+2)=(+5)$.
Since $(+5)\times(-2)=(-10)$, $\therefore\ (-10)\div(-2)=(+5)$.
Since $(-5)\times(+2)=(-10)$, $\therefore\ (-10)\div(+2)=(-5)$.
Since $(-5)\times(-2)=(+10)$, $\therefore\ (+10)\div(-2)=(-5)$.

We therefore make the following **rule of signs** :

$$(+\mathbf{a}) \div (+\mathbf{b}) = \left(+\frac{\mathbf{a}}{\mathbf{b}}\right) = \frac{\mathbf{a}}{\mathbf{b}}\,;\quad (-\mathbf{a}) \div (-\mathbf{b}) = \left(+\frac{\mathbf{a}}{\mathbf{b}}\right) = \frac{\mathbf{a}}{\mathbf{b}}.$$

$$(+\mathbf{a}) \div (-\mathbf{b}) = \left(-\frac{\mathbf{a}}{\mathbf{b}}\right) = -\frac{\mathbf{a}}{\mathbf{b}}\,;\quad (-\mathbf{a}) \div (+\mathbf{b}) = \left(-\frac{\mathbf{a}}{\mathbf{b}}\right) = -\frac{\mathbf{a}}{\mathbf{b}}.$$

provided that b is not zero.

Further, if any number is multiplied by 0, the product is 0. Also, if 0 is divided by any number which is itself not zero, the quotient is 0 ; but we shall never speak of dividing a number by 0.

The rules of sign given above may be stated as follows :

In multiplication and division of one directed number by another, like signs give a positive sign, and unlike signs give a negative sign.

Square Roots

From the rule of signs, we see that

$$(+a) \times (+a) = (+a^2) = a^2 \text{ and } (-a) \times (-a) = (+a^2) = a^2.$$

This shows that a positive number, a^2, has two square roots, $(+a)$ and $(-a)$; these are usually written in the form, $\pm a$. The symbol $\sqrt{(+a^2)}$ or $\sqrt{a^2}$ is used to represent the *positive* square root.

For example, if $x^2 = 9$, then $x = \pm 3$; but $\sqrt{9} = 3$.

There is no square root of a negative number.

EXERCISE VII. c

1. A retired tradesman finds that his bank balance is being diminished at the steady rate of £60 a year. At present he has £300 in the bank. At the end of n years his balance will be £B. Prove that $\mathrm{B} = (+300) + (-60) \times n$.

(i) What was his balance 2 years ago, and what will it be in 3 years' time ?

(ii) What is B when $n = +3,\ -3,\ +5,\ +7$? Interpret the answers.

(iii) What is n when $\mathrm{B} = +60,\ +420,\ -60$? Interpret the answers.

2. A man pays into a bank £10 on the last day of each month. On June 1st, he has £90 in the bank. Take this as zero date and measure the time in months. How would you represent Sept. 1,

April 1, in the same year? Represent by directed numbers, without simplifying, his balance on Sept. 1 and April 1, in the same year, if he draws nothing out of the bank?

What are the values of $(+10) \times (+5)$ and $(+10) \times (-4)$?

3. Repeat No. 2, supposing that the man draws £10 out of the bank on the last day of each month, instead of paying it in?

What are the values of $(-10) \times (+5)$ and $(-10) \times (-4)$?

4. Fig. 120 represents a car travelling northwards and passing A at midnight, zero hour. Represent by directed numbers the distance the car is north of A at 6 minutes past zero hour and at 8 minutes before zero hour. *Afterwards*, simplify the expressions.

FIG. 120.

5. Suppose that in No. 4, the car is travelling $\frac{1}{2}$ mi. per min. southwards; how can you represent its velocity northwards? Represent by directed numbers the distance the car is north of A at 10 minutes past zero hour and at 4 minutes before zero hour. Then simplify the expressions.

6. A farmer says that he loses £100 every year. At the present moment his capital is £4500. Find a formula for his capital in n years' time. What is the meaning and the result if for n we write (i) $+3$, (ii) -4?

7. Repeat No. 6, supposing that the farmer is really gaining £100 every year.

8. A train is travelling eastwards at v feet per sec., see Fig. 121;

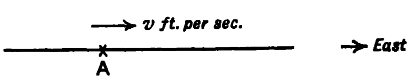


FIG. 121.

at a time t seconds after passing A it is s feet east of A; show that $s = vt$. Evaluate s and interpret the data and results, if

(i) $v = +50,\ t = +10$; (ii) $v = +50,\ t = -10$;
(iii) $v = -50,\ t = +10$; (iv) $v = -50,\ t = -10$.

9. A man looking out of a top-storey window A sees a stone pass the window moving vertically upwards, see Fig. 122; t seconds after passing A, the stone is s ft. above A, where $s = t(+24) + t^2(-16)$. Evaluate s and interpret the data if t equals (i) $-\frac{1}{2}$; (ii) 0; (iii) $+\frac{1}{2}$; (iv) $+1$; (v) $+\frac{3}{2}$; (vi) $+2$.

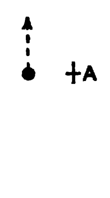


FIG. 122.

EXERCISE VII. d

What are the values of :

1. $(-2)\times(+3)$.
2. $(-3)\times(-4)$.
3. $(+5)\times(-2)$.
4. $(+5)\times(+1)$.
5. $(-3)\times(+5)$.
6. $(-2)\times(-1)$.
7. $(+6)\div(-2)$.
8. $(-12)\div(-4)$.
9. $(-8)\div(+2)$.
10. $(-6)\div(-6)$.
11. $(-6)\div(+1)$.
12. $(+8)\div(-1)$.
13. $(-3)^2$.
14. $(-1)^3$.
15. $(-3)(-4)\div(-6)$.
16. $(-4)\times 2$.
17. $0\times(-5)$.
18. $0\div(-3)$.
19. $\dfrac{(-6)}{(+3)}$.
20. $\dfrac{(-8)}{(-1)}$.
21. $\dfrac{(+4)}{(-1)}$.
22. $(+2t)(-t)$.
23. $(-3r)(-2r)$.
24. $(-s)(+3s)$.
25. $(-4a^2)(+2)$.
26. $(-xy)(-1)$.
27. $(+3y)(-2z)$.
28. $(-6b^2)\div(+2b)$.
29. $(-12c^2)\div(-3c)$.
30. $(+10p^2)\div(-5p)$.
31. $(-12xy)\div(-4)$.
32. $(+5z^2)\div(-1)$.
33. $(-4t^2)\div(-4t)$.
34. $(+6ab)\div(-ab)$.
35. $(-4x^6)\div(2x^3)$.
36. $(-bc^2)\times(-b^2c)$.
37. $(-3a)\times(3bc)$.
38. $0\times(-2xy)$.
39. $0\div(-2x^2)$.
40. $(+pq)\div(-pq)$.
41. $(-r)(+r)$.
42. $(-st)\div(-st)$.

If $a=(+2)$, $b=(-4)$, $p=(+1)$, $q=(-1)$, $r=0$, find the values of the following :

43. $a+b$.
44. $a-b$.
45. $p+q$.
46. $2a+b$.
47. $p-q$.
48. $p+b$.
49. $r-q$.
50. $2q-b$.
51. $3q-p$.
52. $2b-r$.
53. $3a+2b$.
54. $2(p-b)$.
55. ab.
56. pq.
57. br.
58. bpq.
59. b^2.
60. q^3.
61. $2bq$.
62. $3ar$.
63. $\dfrac{b}{a}$.
64. $\dfrac{p}{q}$.
65. $\dfrac{r}{b}$.
66. $\dfrac{pq}{b}$.

Brackets

The object of using brackets in such cases as $(+3)$, (-4), is to distinguish between the positive number " plus three " and the order " add three " or between the negative number " minus four " and the order " subtract four." For simplicity, we write these numbers as 3, -4, if it is unnecessary to emphasise the distinction.

Brackets are also used to bind directed numbers together and may be removed by precisely the same rules as are used for signless numbers, pp. 81, 83.

Example 1. Write more shortly:

(i) $+(-2a)$; (ii) $-3(+2b)$; (iii) $+(-2)(-3c)$.

(i) $+(-2a)$ means " add the number $(-2a)$ "; this has the same result as " subtract $(+2a)$ "; we write it $\mathbf{-2a}$.

(ii) $-3(+2b) = -(+6b) = \mathbf{-6b}$.

(iii) $+(-2)(-3c) = +(+6c) = \mathbf{6c}$.

Example 2. Simplify $(+3a) - 4\{(-2b) - (+3c)\}$.

The expression $= (+3a) - 4(-2b) + 4(+3c)$

$= (+3a) - (-8b) + (+12c) = \mathbf{3a + 8b + 12c}$.

Example 3. Simplify $2x(x+y) - 3y(x-2y)$.

The expression $= (2x^2 + 2xy) - (3xy - 6y^2)$

$= 2x^2 + 2xy - 3xy + 6y^2 = \mathbf{2x^2 - xy + 6y^2}$.

Example 4. Simplify $(-6a^2b) \div (-4ab^2)$.

$$\frac{-6a^2b}{-4ab^2} = \frac{6a^2b}{4ab^2} = \frac{2ab \times 3a}{2ab \times 2b} = \frac{\mathbf{3a}}{\mathbf{2b}}.$$

Example 5. Fill in the blanks in:

(i) $a - b = -(\ldots\ldots)$; (ii) $\dfrac{b-a}{c-d} = \dfrac{a-b}{\ldots\ldots}$.

(i) $a - b = -(-a) - (+b) = -(-a+b) = -(b-a)$.

(ii) $\dfrac{b-a}{c-d} = \dfrac{-(b-a)}{-(c-d)} = \dfrac{-b+a}{-c+d} = \dfrac{a-b}{d-c}$.

Note. It is customary to arrange an expression so that it starts with a positive sign, thus we write $b - a$ rather than $-a + b$.

EXERCISE VII. e

Write more shortly:

1. $+2(-5a)$. **2.** $+3(+4b)$. **3.** $+(-2c)$.

4. $-(-d)$. **5.** $(-4)(+2e)$. **6.** $-(+3)(-3x)$.

7. $-3(+5y)$. **8.** $+(-4)(-2z)$.

Simplify the following:

9. $(-2a) - (-3a) - (+2a)$. **10.** $(+4p) + (-3q) - (-2r)$.

11. $-3(p - 2q) + p$. **12.** $-(a - b) - c$.

13. $2r - 3\{(-s) - (-2t)\}$. **14.** $3x + 2\{(+3y) - (-z)\}$.

15. $2(b-c)-3(b+c)$.

16. $3(p+q)+5(p-q)$.

17. $2x-y-(x+y)$.

18. $2c-(c-d)-3d$.

19. $-5(n-p)+3(p-n)$.

20. $0-2(c-a)$.

21. $-(y-z)-(z-x)-0$.

22. $a(b-c)-c(b-a)$.

23. $2(1-x)-3x(1-x)$.

24. $-x(x-y)+y(y-x)$.

25. $3x^2-5x-2(1-x+x^2)$.

26. $(3a-2b-c)-(a-b+c)$.

27. $(p-q)(-1)-3(q-p)$.

28. $(y^2-3y-2)(-y)$.

Copy and complete the following :

29. $2a-6b=(+2)\{\ldots\ldots\}=(-2)\{\ldots\ldots\}$.

30. $-p+3q=(-1)\{\ldots\ldots\}=+\{\ldots\ldots\}$.

31. $r+s=-(\ldots\ldots)$.

32. $x-y-z=x-(\ldots\ldots)$.

33. $p-4q+4r=p-4(\ldots\ldots)$.

34. $b+3c-3d=b+3(\ldots\ldots)$.

35. $\dfrac{a}{-b}=\dfrac{-a}{\ldots\ldots}$.

36. $\dfrac{-c}{-d}=\dfrac{c}{\ldots\ldots}$.

37. $x^2=(-x)(\ldots\ldots)$.

38. $(-x)^2=x\times(\ldots\ldots)$.

39. $\dfrac{x}{-y}=\ldots\ldots\dfrac{x}{y}$.

40. $\dfrac{a(-b)}{(-c)(d)}=\ldots\ldots$

41. $\dfrac{(-p)(-q)}{r(-s)}=\ldots\ldots$

42. $-\dfrac{3}{-b}=\ldots\ldots$

43. $\dfrac{r-s}{b-a}=\dfrac{\ldots\ldots}{a-b}$.

44. $\dfrac{c\quad d}{e+f}=-\ldots\ldots$

45. $(-a-b)(-a-b)=(a+b)(\ldots\ldots)$.

46. $(d-c)(d+c)=(c-d)(\ldots\ldots)$.

Simplify the following :

47. $y^2-(-y)^2$.

48. $(a^2b^2)\div(-b)$.

49 $(2c)^2\div(-c)^2$.

50. $-(3rs)^2$.

51. $(-2pq)^2$.

52. $8b^3\div(-4b)$.

53. $(-2t)^3$.

54. $\left(-\dfrac{a}{b}\right)(-2ab)$.

55. $\left(\dfrac{3}{p}\right)(-p^3)$.

56. $\dfrac{3yz}{-y}$.

57. $\dfrac{-p^2}{-pq}$.

58. $\dfrac{-b}{bc}$.

59. $\dfrac{a-b}{b-a}$.

60. $\dfrac{a+b}{-a-b}$.

61. $\dfrac{(a-b)^2}{(b-a)^2}$.

62. What can you say about x if (i) $x^2=16$, (ii) $x^2=1$?

63. Solve the equation, $2y^2-1=49$.

64. What can you say about n if $n=\frac{36}{n}$?

65. If $xy=48$ and $y=3x$, find x and y.

[*Note. For additional drill-examples, see Ex. E.P.* 10, 11, 12, *pp.* 141-143.]

In the previous oxamples, brackets have been used for performing addition, subtraction, multiplication and division. The working may, however, be arranged as in Arithmetic.

Example 6. Add $3y-x-5z$ to $3x-3y+z$.

Arrange the terms of the two expressions in similar orders.

$$\begin{array}{r} -x+3y-5z \\ \underline{3x-3y+\ \ z} \\ \underline{\mathbf{2x}\qquad\ \ -\mathbf{4z}} \end{array}$$

Example 7. Subtract $3y-x-5z$ from $3x-3y+z$.

If we use brackets, we write

$$(3x-3y+z)-(3y-x-5z)=3x-3y+z-3y+x+5z$$
$$=\underline{\mathbf{4x-6y+6z}}$$

This shows that if the expressions are set down as in Arithmetic, the result is obtained by changing the sign of each term in the lower line (mentally) and then adding

$$\begin{array}{r} 3x-3y+\ \ z \\ \underline{-x+3y-5z} \\ \underline{\mathbf{4x-6y+6z}} \end{array}$$

NOTE. This mechanical method of subtraction is used in the solution of simultaneous equations by "addition and subtraction" (see p. 149), also in long division. Considerable drill is necessary if it is desired to make the process automatic.

Example 8. Divide $12a^3b-6a^2b^2+18ab^3$ by $-6ab$.

$$-6ab\ )\ \underline{12a^3b-6a^2b^2+18ab^3}$$
$$\mathbf{-2a^2\ +\ ab\ -\ 3b^2}$$

EXERCISE VII. f

Add :

1. $3a - b$ and $3b - a$.
2. $c - 7 - c^2$ and $3 - c + c^2$.
3. $2d^2 - 1$ and $-d^2$.
4. $2r + 3s - t$ and $2s - r - t$.
5. $p + q - r$ and $q + r - p$.
6. $c - 3a + 2b$ and $b - a - 3c$.
7. $1 - (x + y)$ and $y - (x - 1)$.
8. $ab - a^2 + b^2$ and $a(a + b)$.
9. $2r(r - s)$ and $3s(s - r)$.
10. $2(3t - 1 - t^2)$ and $-3(2t - t^2)$.
11. $3b - 2c - d$, $c - 3d - b$, $2d - 3c$.
12. $3l - 2m - n$, $3m - 2n - l$, $3n - 2l - m$.
13. $-x - y$, $-y - z$, $-z - x$, $-x - y - z$.
14. $1 - x - 3x^2$, $2x - 5 - x^2$, $2x^2 - x + 2$.

Subtract :

15. $2a + b$ from $3a + 3b$.
16. $2c - d$ from $2d - c$.
17. $x^2 + y^2$ from $x^2 - y^2$.
18. $r^2 - 1$ from $2r^2$.
19. $2t$ from $1 + t$.
20. $3a$ from $a - b$.
21. $a - b + c$ from $b + c - a$.
22. $2p - 3q - r$ from $q - p + r$.
23. $3x + y + 1$ from $4x - y - 1$.
24. $2c - d - e$ from $d - 2e$.
25. $3x^2 - 5x - 6$ from x^2.
26. $3(2t - 1)$ from $4(3 - t)$.
27. $1 + d - d^2$ from 0.
28. $2r - s - t$ from $-3s - 5t$.

Multiply :

29. $x + y$ by $-z$.
30. $b - c$ by $-a$.
31. $t - 1$ by $-t$.
32. $r - 2s$ by $-3s$.
33. $2c - 3c^2$ by 0.
34. $3y - y^2$ by -1

Divide :

35. $a^2 - ab$ by $-a$.
36. $p^2 + pq$ by $-p$.
37. $x - y$ by -1.
38. $6r^3 - 4r^2$ by $-2r$.
39. $s^4 + s^2$ by $-s^2$.
40. $pq - q^2$ by $-q$.
41. $a^2 - ab + 2ac$ by $-a$.
42. $6x^3 - 8x^2y - 4xy^2$ by $-2x$.

Subtract :

	(i)	(ii)	(iii)	(iv)
43.	$\begin{array}{r} 4x + y \\ \underline{x + 3y} \end{array}$	$\begin{array}{r} 5x - 3y \\ \underline{3x + 2y} \end{array}$	$\begin{array}{r} 7x + 4y \\ \underline{3x - 5y} \end{array}$	$\begin{array}{r} 6x - 4y \\ \underline{4x - 5y} \end{array}$
44.	$\begin{array}{r} 2a - 6b \\ \underline{a - 2b} \end{array}$	$\begin{array}{r} 5a + 2b \\ \underline{3a - 2b} \end{array}$	$\begin{array}{r} 4a + b \\ \underline{a + 4b} \end{array}$	$\begin{array}{r} 5a - 3b \\ \underline{a + b} \end{array}$

45. $\begin{array}{r} 3c+2d \\ c+2d \\ \hline \end{array}$ $\quad$ $\begin{array}{r} 2c-5d \\ 5c-6d \\ \hline \end{array}$ $\quad$ $\begin{array}{r} c-3d \\ 4c-3d \\ \hline \end{array}$ $\quad$ $\begin{array}{r} 4c+3d \\ 6c-3d \\ \hline \end{array}$

46. $\begin{array}{l} 2p \\ 3p-q \\ \hline \end{array}$ $\quad$ $\begin{array}{r} 3q \\ p+5q \\ \hline \end{array}$ $\quad$ $\begin{array}{r} p-q \\ q \\ \hline \end{array}$ $\quad$ $\begin{array}{l} 3p+2q \\ 4p \\ \hline \end{array}$

47. $2x-y+1$ from $x+2y-1$. **48.** $x+y+6$ from $3x-2y+4$.

49. $3a-4b-3$ from $a+4b+1$. **50.** $2a+b-4$ from $a-3b-3$.

51. $c+3d+2$ from $c-d-1$. **52.** $5-2c-d$ from $3-c-2d$.

Directed Numbers in Problems and Equations

Example 9. Solve the equation $1-2\frac{1}{2}p=7$.

$$1-2\tfrac{1}{2}p=7\ ; \qquad \therefore\ -2\tfrac{1}{2}p=7-1=6.$$

Multiply each side by -2.

$$\therefore\ -\frac{5p}{2}\times(-2)=6\times(-2)\ ;$$

$$\therefore\ 5p=-12\ ;$$

$$\therefore\ p=-\frac{12}{5}=-2{\cdot}4.$$

Example 10. Two express trains A and B travelling at 42 miles an hour and 54 miles an hour respectively are both proceeding due East. When A is 15 miles east of a town T, B is 21 miles east of T. How far from T is the place P where B passed A ?

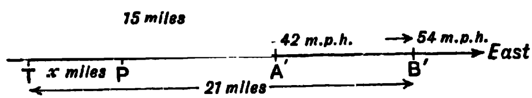


FIG. 123.

Let P be x miles east of T.

Represent the data on a figure.

A travels $(15-x)$ miles beyond P, in the same time that B travels $(21-x)$ miles beyond P, because they pass P at the same time.

A, travelling at 42 m.p.h., goes $(15-x)$ miles in $\frac{15-x}{42}$ hours;

B, travelling at 54 m.p.h., goes $(21-x)$ miles in $\frac{21-x}{54}$ hours;

$$\therefore\ \frac{15-x}{42}=\frac{21-x}{54}.$$

L.C.M. of 42 and 54 is $6 \times 7 \times 9$; $\therefore\ 9(15-x)=7(21-x)$;

$$\therefore\ 135-9x=147-7x;\ \therefore\ -2x=12;$$

$$\therefore\ 2x=-12;\ \therefore\ x=-6.$$

$$\therefore\ \text{P is } (-6) \text{ miles east of T.}$$

This means that B passed A at a place P, 6 *miles west* of T, assuming that neither express train stopped at T.

EXERCISE VII. g

1. The present ages of A and B are 21 and 35; in n years' time, B will be twice as old as A. Find n and interpret the answer.

2. Two large kettles are being heated, A on a gas stove and B on a primus; at t minutes past eleven, the temperatures in degrees Centigrade of A and B are $30+2t$ and $48+5t$. At what time is the water at the same temperature in the two kettles? Each kettle was filled with water at 14° C.? At what times were the kettles put on the stoves?

3. Can you find six consecutive odd numbers whose sum is 12?

4. What number must be added to both numerator and denominator of the fraction $\frac{17}{25}$, so that the result is equal to $\frac{3}{7}$?

5. The heights of A and B above sea-level are a feet and b feet; and the height of A above C is equal to the height of C above B. Find the height of C. Interpret the answer when (i) $a=100$, $b=-200$; (ii) $a=50$, $b=-50$.

6. If C° Centigrade is the same temperature as F° Fahrenheit, $C=\frac{5}{9}(F-32)$. Express 0° Fahrenheit in Centigrade. What temperature is represented by the same number on the two scales?

7. Two cars P, Q are travelling in the same direction along a road at u and v miles per hour respectively. At noon, Q is s miles ahead of P. At what time will P pass Q? What does your answer become if $u=20$, $v=24$, $s=2$, and what does it mean?

Solve the equations:

8. $1\frac{1}{2}p=-12.$

9. $-1\frac{3}{4}q=35.$

10. $7=-2t.$

11. $\frac{l}{4}=\frac{l+1}{3}.$

12. $\frac{x}{3}-\frac{x}{2}=1.$

13. $3(1-k)-5(2-k)=k-12.$

14. $\frac{1}{y}=-3.$

15. $A-\frac{5}{3}A=5.$

16. 15 years ago a father was three times the age of his son and 19 years ago he was four times the age of his son. How old are they now? In how many years' time will he be twice his son's age?

17. The marks obtained in an examination ran from 24 to 84; these were then scaled so as to run from 0 to 100. What was the scaled mark corresponding to n marks for the paper? A boy, who did the paper afterwards, obtained 15 marks for it, what would his mark become according to the same scale?

18. A small body P of weight W lb. is just displaced from the highest point A of a fixed sphere, whose diameter AB is d feet. In Fig. 124, AN represents the *vertical* distance P has fallen. When

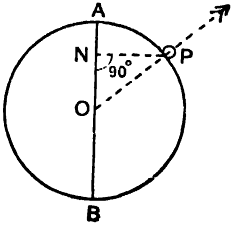


FIG. 124.

AN $= h$ feet, it can be proved that the pressure of the sphere on the body outwards from the centre O is $\frac{W}{d}(d - 6h)$ lb.-wt. Evaluate this when P is level with O and interpret the result.

Find the depth of P below A when it leaves the surface of the sphere.

CHAPTER VIII

SIMPLE EQUATIONS AND PROBLEMS

In the following example, the steps enclosed in square brackets can be omitted as soon as the process is understood.

Example 1. Solve the equation

$$1-\frac{1}{6}(t+5)=\frac{2t+7}{3}-\frac{t-10}{4}.$$

[The L.C.M. of 6, 3, 4 is 12.] Multiply each side by 12.

$$\left[\therefore\ 12-\frac{12}{6}(t+5)=\frac{12(2t+7)}{3}-\frac{12(t-10)}{4}\right]$$

$$\therefore\ 12-2(t+5)=4(2t+7)-3(t-10);$$

$$[\therefore\ 12-(2t+10)=8t+28-(3t-30)]$$

$$\therefore\ 12-2t-10=8t+28-3t+30;$$

$$\therefore\ 2-2t=5t+58;\ \therefore\ -2t-5t=58-2;$$

$$\therefore\ -7t=56.$$

Multiply each side by -1, $\therefore\ 7t=-56.$

$$\therefore\ t=-8.$$

Check: If $t=-8$,

$$\text{left side}=1-\tfrac{1}{6}(-8+5)=1-\tfrac{1}{6}(-3)=1+\tfrac{3}{6}=1\tfrac{1}{2}.$$

$$\text{right side}=\frac{-16+7}{3}-\frac{-8-10}{4}=\frac{-9}{3}-\frac{-18}{4}=-3+4\tfrac{1}{2}=1\tfrac{1}{2}.$$

$$\therefore\ \text{if } t=-8,\ \text{left side}=\text{right side}.$$

EXERCISE VIII. a

Solve the following equations:

1. $5(x-2)-3(x-1)=-1.$
2. $3(6-a)-4(a+8)=0.$
3. $p+1=2(p-3)-3(p-1).$
4. $2(y-1)-3(3+y)=5+y.$
5. $2b-(1-2b)=5-3(1+b).$
6. $4-4(k-5)=2(2-k)-6.$
7. $0=2(n+1)-5(n-5).$
8. $4(x-1)-3(x-2)=4-5x.$
9. $4r-\frac{1}{5}(r+4)=41.$
10. $\frac{1}{2}(z-1)-\frac{1}{4}(3-z)=2.$
11. $\dfrac{2x-1}{5}=3+\dfrac{1-3x}{4}.$
12. $\dfrac{2t-1}{5}-\dfrac{1+t}{2}=1.$
13. $3(2a-5)-5(4-a)=\frac{1}{2}a+7.$
14. $\dfrac{1}{2y}-\dfrac{1}{3y}=\dfrac{1}{5}.$

15. $\frac{1}{3}(4R+1)-\frac{1}{4}(2-3R)=\frac{1}{6}(26-R)$.

16. $\frac{5}{d}+\frac{1}{3}=\frac{3}{4}$.

17. $\frac{1-v}{2}=\frac{3+v}{3}-\frac{9+v}{4}$.

18. $\frac{1}{6}(4x-3)-\frac{1}{8}(5x-3)=1$.

19. $c-1-\frac{1}{2}(c-2)+\frac{1}{3}(c-3)=0$.

20. $\frac{p+1}{3}-\frac{p-1}{2}=1+\frac{2p}{3}$.

21. $\frac{7}{3u}-5=\frac{1}{2}-\frac{5}{u}$.

22. $\frac{2y-1}{5}-\frac{3y+1}{2}=\frac{2}{5}$.

23. $\frac{3}{2x}-4=3-\frac{9}{x}$.

24. $\frac{t+1}{2}+\frac{t+2}{3}=14-\frac{t-5}{4}$.

25. $\frac{h}{3}+\frac{h}{4}=55-\frac{h+40}{5}$.

26. $\frac{1}{7}(2a-1)+12=\frac{7a-2}{5}$.

27. $\frac{2y-1}{3}+\frac{4y+1}{5}+\frac{12-3y}{4}=0$.

28. $\frac{3l+4}{5}-\frac{7l-3}{2}=\frac{l-16}{4}$.

29. $0{\cdot}6x=0{\cdot}72$.

30. $1{\cdot}8y=-0{\cdot}63$.

31. $1{\cdot}2x-0{\cdot}5x=4{\cdot}2$.

32. $1{\cdot}4t+0{\cdot}6t=0{\cdot}7$.

33. $0{\cdot}3(6-n)-0{\cdot}4(n+8)=0$.

34. $\frac{1}{2}(x+1{\cdot}7)-\frac{1}{3}(x-2{\cdot}3)=1$.

35. $1{\cdot}4(a-3)-0{\cdot}4(2a-1)=-0{\cdot}2$.

36. $\frac{1}{3}(1+{\cdot}2y)-\frac{1}{7}(2-{\cdot}3y)=0{\cdot}19$.

37. $0{\cdot}01(p-5)=0{\cdot}24(3-p)$.

38. $\frac{t-1}{0{\cdot}5}+\frac{2t+1}{0{\cdot}75}=18$.

39. $n(2n-1)-3(5+n)=2n^2+1$.

40. $r^2-r(5-r)=6-2r(1-r)$.

41. $3\left(\frac{a}{4}-1\right)-2\left(\frac{a}{3}+1\right)=\frac{a}{6}-\frac{2}{3}$.

42. $\frac{3z-1}{5}-\frac{1+z}{2}=3-\frac{z-1}{4}$.

43. $\frac{2\frac{1}{3}(x+7)}{4}-\frac{2\frac{1}{2}(x+1)}{3}=2$.

44. $\frac{y+5}{2\frac{1}{2}}-\frac{y-1}{3\frac{1}{3}}=1$.

[*Note. For additional drill-examples, see Exercise E.P.* 13, *p.* 144.]

Formulae and Equations

Example 2. If a man is photographed, when standing u inches from the lens of a camera, focal length f inches, the plate should be v inches from the lens, where $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$.

Find the distance of the plate from a lens of focal length $4\frac{1}{2}$ in., if the man is 6 feet from the lens.

Here, $$f=4\tfrac{1}{2} \text{ and } u=6\times 12.$$

$$\therefore\ \frac{1}{72}+\frac{1}{v}=\frac{1}{4\frac{1}{2}}=\frac{2}{9}.$$

Either, multiply each side by $72v$,

then $$v+72=16v\ ;\quad \therefore\ 15v=72\ ;$$

$$\therefore\ v=\frac{72}{15}=4\cdot 8.$$

Or, $$\frac{1}{v}=\frac{2}{9}-\frac{1}{72}=\frac{16-1}{72}=\frac{15}{72}.$$

$$\therefore\ v=\frac{72}{15}=4\cdot 8.$$

$\therefore$ the distance of the plate from the lens is **4·8 inches.**

EXERCISE VIII. b

1. The area of a triangle is given by the formula, $A=\frac{1}{2}h\,.\,b$; find b if $A=6\frac{3}{4}$ and $h=1\frac{1}{8}$.

2. The area of a trapezium is given by the formula, $A=\frac{1}{2}h(a+b)$ find b if $A=5\cdot 4$, $a=1\cdot 7$, $h=3\cdot 6$.

3. If A exceeds P by R per cent., then $A=P\left(1+\frac{R}{100}\right)$; find R if $P=96$, $A=120$.

4. If F° Fahrenheit is the same temperature as C° Centigrade, then $F=32+\frac{9C}{5}$; find C if $F=5$.

5. The pressure p gm. per sq. cm., the volume v cu. cm., and the temperature t degrees Centigrade of a certain mass of gas are connected by the formula $\frac{pv}{273+t}=9\cdot 6$. Find t if $p=16$, $v=180$.

6. A test made on a differential wheel and axle showed that a load of W kg. could be raised by an effort of P kg., where $P=0\cdot 075W+0\cdot 05$. What load could be raised by an effort of 2 kg. ?

7. If the velocity of a body, sliding downhill, increases steadily from u ft. per sec. to v ft. per sec., in t seconds, the distance, s feet, it moves in that time is given by $s=\frac{u+v}{2}\,.\,t$.

Find v if $s=84$, $u=9$, $t=3\frac{1}{2}$

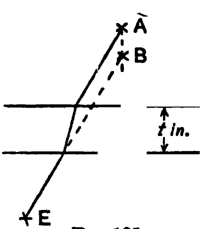


FIG. 125.

8. If an object A, see Fig. 125, is viewed from E through a transparent plate of thickness, t inches, and refractive index μ, it appears to be at B, which is d inches nearer the plate than it really is, where $d=t\,.\,\frac{\mu-1}{\mu}$. Find μ if $d=0\cdot 25$ when $t=0\cdot 8$.

9. If the perimeter of a semicircular area of diameter d in.

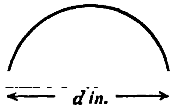


FIG. 126.

is p in., see Fig. 126, then $p = d + \frac{1}{2}\pi d$, where π may be taken as $\frac{22}{7}$; find d if $p = 4 \cdot 5$.

10. After an examination the marks are scaled so that a boy who obtained n marks for the paper receives N marks, where $\frac{N}{100} = \frac{n - 17}{72 - 17}$. What is n if N equals (i) 40, (ii) 100, (iii) 0 ?

11. The velocity-ratio of a differential pulley system, see Fig. 127, is n where $n = \frac{2R}{R - r}$, R and r being the radii of the grooves in the pulley. Find R if $n = 18$ and $r = 2\frac{2}{3}$.

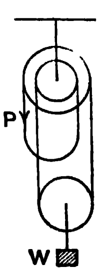


FIG. 127.

12 The area of the total surface of a solid circular cylinder of height h in. and base-radius r in. is A sq. in., where $A = 2\pi r(r + h)$ and π may be taken as $\frac{22}{7}$. If $A = 5 \cdot 5$ and $r = 0 \cdot 2$, find h.

13. If a bath can be filled by one tap in p minutes and by another tap in q minutes, it can be filled by both taps together in t minutes, where $\frac{1}{t} = \frac{1}{p} + \frac{1}{q}$. Find p if $t = 4\frac{1}{2}$ and $q = 7\frac{1}{2}$.

14. From the formula $s = \frac{n}{2}[2a + (n - 1)d]$ for the sum of an arithmetical progression, find d if $a = 9$, $n = 16$, $s = 48$.

15. If $\frac{x}{2} - 1$ is twice as large as $\frac{x}{3} - 4$, find the value of each.

16. If $\frac{y^2 - 1}{3} + \frac{2y^2 + 7}{5} = \frac{y^2 + 4}{2}$, what is the value of y^2 ? What can you say about the value of y ?

Problems

When *solving* a problem,

(i) If possible, make a rough diagram and show the data on it.

(ii) Choose a letter for some unknown number the problem involves, and state precisely what this letter represents.

(iii) Re-write the question, using the letter you have chosen for the unknown to make the statement of the problem more detailed.

When *checking* the answer to a problem,

Use the actual data of the problem. It is not sufficient to check by substituting in the equation, because your equation may be wrong.

Example 3. I walk at $3\frac{1}{2}$ miles an hour from my house to a town, by a path through the fields. After waiting 20 minutes, I return in a bus travelling at $10\frac{1}{2}$ miles an hour. If the road adds another mile to the journey and if the total time taken is 2 hours, find the distance by road from my house to the town.

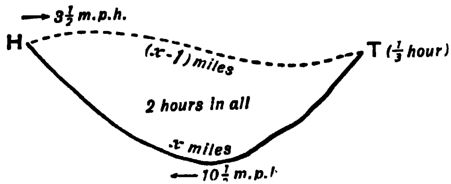


FIG. 128.

Let the distance *by road* from my house to the town be x miles.

Then the distance by the path is $(x-1)$ miles.

[Mark the facts on the diagram and re-write the question.]

I walk $(x-1)$ miles at $3\frac{1}{2}$ miles an hour from H to T, wait for $\frac{1}{3}$ hour at T, and ride back x miles at $10\frac{1}{2}$ miles per hour; the total time is 2 hours.

$\therefore$ the journey from H to T takes $\dfrac{x-1}{3\frac{1}{2}}$ hours,

and the journey back takes $\dfrac{x}{10\frac{1}{2}}$ hours;

$$\therefore \frac{x-1}{3\frac{1}{2}}+\frac{1}{3}+\frac{x}{10\frac{1}{2}}=2.$$

$$\left[\frac{x-1}{3\frac{1}{2}}=\frac{2(x-1)}{2\times 3\frac{1}{2}}=\frac{2(x-1)}{7};\quad \frac{x}{10\frac{1}{2}}=\frac{2x}{2\times 10\frac{1}{2}}=\frac{2x}{21}\right].$$

$$\therefore \frac{2(x-1)}{7}+\frac{1}{3}+\frac{2x}{21}=2.$$

Multiply each side by 21, $\therefore\ 6(x-1)+7+2x=42.$

$$\therefore\ 6x-6+7+2x=42;\quad \therefore\ 8x=41;$$

$$\therefore\ x=\frac{41}{8}=5\tfrac{1}{8}.$$

$\therefore$ **the distance by road from the house to the town is $5\frac{1}{8}$ miles.**

Check : The distance by the path is $4\frac{1}{8}$ miles.

To walk $4\frac{1}{8}$ mi. at $3\frac{1}{2}$ m.p.h. takes $4\frac{1}{8} \div 3\frac{1}{2}$ hours $= \frac{33}{8} \times \frac{2}{7}$ hrs.

$$= \frac{33}{28} \text{ hours.}$$

To ride $5\frac{1}{8}$ mi. at $10\frac{1}{2}$ m.p.h. takes $5\frac{1}{8} \div 10\frac{1}{2}$ hours $= \frac{41}{8} \times \frac{2}{21}$ hrs.

$$= \frac{41}{84} \text{ hours.}$$

$$\text{Total time} = \left(\frac{33}{28} + \frac{1}{3} + \frac{41}{84}\right) \text{hours} = \frac{99 + 28 + 41}{84} \text{hours} = 2 \text{ hours.}$$

EXERCISE VIII. c

1. Find a value of n for which the fraction $\dfrac{5n+2}{7n+1}$ reduces to $\frac{3}{4}$.

2. Fig. 129 represents a cyclist riding from A to B and back again ; the double journey takes 5 hours. Find x.

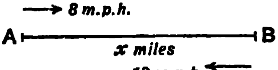


FIG. 129.

3. Fig. 130 gives the lengths of three sides of a rectangle in inches. Find numerical expressions for the length of the fourth side and the perimeter.

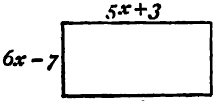


FIG. 130.

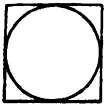


FIG. 131.

4. A kettle of water is placed on a stove ; after t minutes, its temperature is $(16+5t)$ degrees Centigrade. How long does it take to boil ?

What was the temperature at the beginning ?

5. A piece of wire 20 inches long is cut into two pieces, one of which is bent into a circle and the other forms the square enclosing it, see Fig. 131. What is the diameter of the circle ? Take $\pi = \frac{22}{7}$.

6. If N + (40 per cent. of N) equals 84, find N.

7. A man rows upstream at 3 miles an hour and back to the same place at 5 miles an hour; he takes 48 minutes altogether. How far upstream did he go?

8. Fig. 132 represents a cyclist leaving A at the same moment as a pedestrian leaves B. How far from A do they meet?

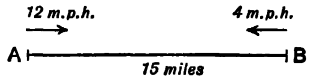


FIG. 132.

9. A river flows at 3 miles an hour. What is the speed through the water of a boat that can go downstream twice as fast as upstream?

10. If C° Centigrade represents the same temperature as F° Fahrenheit, $C = \frac{5}{9}(F - 32)$. What is the temperature which is recorded by a number twice as large on the Fahrenheit scale as on the Centigrade scale?

11. Fig. 133 gives the lengths of the sides of a triangle in inches. If the triangle is isosceles, find its perimeter.

12. A table is marked at £P, but there is a discount of 3d. in the shilling for cash; the cash price is £12. What is P?

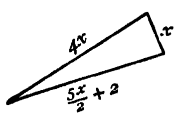


FIG. 133.

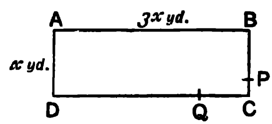


FIG. 134.

13. Fig. 134 represents a rectangular enclosure; $CP = \frac{1}{4}CB$; $CQ = \frac{1}{4}CD$; the path ABP is 12 yards longer than ADQ. Find the length of BC.

14. At a fair, a boy receives 8d. for a hit and pays 3d. for a miss; 24 shots cost him 6d.; how many hits did he score?

15. The "rise" AB, R inches, and the "tread" BC, T inches, of

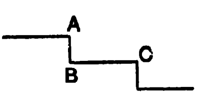


FIG. 135.

a staircase, see Fig. 135, are connected by the formula $R = \frac{1}{2}(24 - T)$. What is the rise, if it equals five-sixths of the tread?

16. A rectangle is l feet long; its perimeter is 5 feet; what is its breadth? Another rectangle is twice as long and half as broad as the first, and its perimeter is 7 feet. Find l.

17. Find a number such that if you add 7 and divide the sum by 5 you will get the same answer as if you had subtracted 1 and then divided by 3.

18. A boy counts 2 marks for each sum he gets right and (-1) mark for each he gets wrong. He does 18 sums and obtains 15 marks. How many sums did he get right?

19. In a factory, the men get 7s. 6d. a day each and the women get 6s. a day each; 200 people are employed and the wages amount to £69 a day. How many men are there?

20. A messenger goes on an errand at 4 miles an hour; 20 minutes later, a boy bicycles after him at 12 miles an hour; how far must he go to overtake the messenger?

21. If n half-crowns and $(2n-1)$ florins make up three guineas, what is n?

22. My train starts in 12 minutes, and the station is 1 mile away. I walk at 4 miles an hour and run at 8 miles an hour. How far must I run?

23. Use Fig. 136 to find two consecutive whole numbers whose squares differ by 37.

24. Find v, if v miles an hour is the same speed as $(v+3\frac{1}{2})$ feet per second.

25. If I walk to the station at 4 m.p.h. to catch a train, I shall have 3 minutes to spare; but if I walk at 3 m.p.h., I shall miss the train by 1 minute. How far off is the station?

26. A man buys 800 bulbs for two guineas, some of them at 25 for a shilling and the rest at 6s. a hundred; how many were there of the more expensive kind?

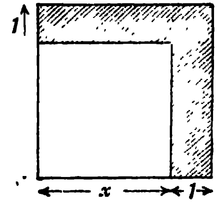


FIG. 136.

27. Fig. 133 gives the lengths of the sides of a triangle in inches. What is the least possible value of x? [In any triangle ABC, $AB + BC > AC$]. What is the greatest possible value of x?

28. I pay no tax on the first £135 of my income, I pay a tax of 2s. in the £ on the next £225, and a tax of 4s. in the £ on the rest. My income after the tax is deducted is £577 10s. What is my gross income?

29. 5 lb. of tea at a certain price is mixed with 4 lb. of tea costing 9d. per lb. more. The average price of the mixture is 2s. 7d. per lb. Find the price of the cheaper kind.

30. A man buys one lot of eggs at 1s. 6d. a dozen and a second lot, which is 3 dozen more than the first lot, at 2s. a dozen; he sells them all at 2s. 6d. a dozen and makes 15s. profit. How many eggs did he buy altogether?

[*Note. For additional examples, see Appendix., Ex. S.* 7, *p.* 291.]

Transformation of Formulae

Most formulae are given or remembered in some standard form. But for the purposes of a particular problem it is often a convenience to have the appropriate formula expressed differently.

Example 4. If the simple interest on £P, lent for T years, at R per cent. per annum, is £I, find the formula for I.

The interest on £100 for 1 year is £R.

$\therefore$ the interest on £P for 1 year is £$\left(R \times \frac{P}{100}\right)$.

$\therefore$ the interest on £P for T years is £$\left(R \times \frac{P}{100} \times T\right)$.

$$\therefore \ £I = £\frac{PRT}{100}; \quad \therefore \ I = \frac{PRT}{100}.$$

I is called the **subject** of this formula.

Now, suppose the problem is expressed in a different form, as follows :

Example 5. If the simple interest on £P, lent for T years, is £I. find the formula for the rate per cent. per annum, R.

Here, we require a formula for R in terms of P, T, I.

$$\frac{PRT}{100} = I.$$

Multiply each side by 100, $\therefore$ PRT = 100 I.

Divide each side by PT, $\therefore \ R = \frac{100 \, I}{PT}$.

This is a formula, whose subject is R. It has been obtained from the simple-interest formula, expressed with I as subject, *by using precisely the same methods as are employed in solving equations.* The process is called **changing the subject of the formula.**

EXERCISE VIII. d

1. What relation connects x and y in Fig. 137 ? Make (i) y,

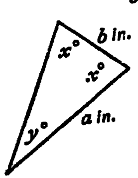


FIG. 137.

(ii) x the subject of the formula.

2. What relation connects a and b in Fig. 138? Make (i) a, (ii) b the subject of the formula.

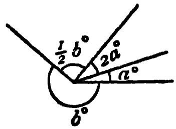


FIG. 138.

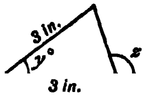


FIG. 139.

3. What relation connects y and z in Fig. 139? Make (i) y, (ii) z the subject of the formula.

4. The perimeter of Fig. 137 is p in.; what formula connects a, b, p? Make a the subject.

5 Fig. 140 represents a rectangular field, of perimeter p yards. Find formulae for the following cases:

(i) Given l, b, make A the subject.
(ii) Given A, l, make b the subject.
(iii) Given p, b, make l the subject.

6. The perimeter of the rectangle in Fig. 140 is p yards; find a formula for A in terms of l, p.

7. The perimeter of the rectangle in Fig. 140 is p yards; find a formula for p in terms of b, A.

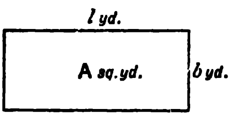


FIG. 140.

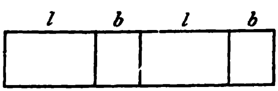


FIG. 141.

8. A hall is l ft. long, b ft. broad, h ft. high; it contains V cu. *yards* of air. Express V in terms of l, b, h and then make h the subject of the formula.

9. If m miles is equivalent to k kilometres, find a formula for k in terms of m. [Take 8 km. $=5$ mi.]

10. Fig. 141 represents the four walls of a room folded out flat, the units are feet. The total area of the walls is A sq. ft. Express A in terms of l, b, h. Then make (i) h, (ii) b the subject of the formula.

11. Interpret geometrically the formula $A=\frac{1}{2}b \cdot h$; make h the subject. Find h if $A=3\frac{1}{2}$, $b=5$.

12. Interpret the formula $C=2\pi r$ and make r the subject.

13. If F° Fahrenheit is the same temperature as C° Centigrade, then $F=32+\frac{9C}{5}$. Make C the subject of the formula. Find the value of C when (i) $F=32$, (ii) $F=212$.

14. Interpret the formula, $A = P + \frac{7PR}{100}$, and make R the subject of the formula.

15. (i) What is the number whose square root is 7 ?

(ii) What is N, if $\sqrt{N} = 8$?

(iii) What is A, if $\sqrt{A} = l$?

16. Make A the subject of the formula,

(i) $\sqrt{A} = 3l$; (ii) $\sqrt{(2A)} = 4b$.

17. Make l the subject of the formula,

(i) $l^2 = S$; (ii) $(l-1)^2 = S$.

18. The edge of a cube is l in. long ; the total area of its surface is A sq. in. ; express A in terms of l, and then make l the subject. What is l if $A = 96$?

19. Interpret the formula, $A = \pi r^2$, and make r the subject.

20. Interpret the formula, $V = \pi r^2 h$, and make (i) h, (ii) r the subject.

21. The area of a trapezium is given by the formula, $A = \frac{1}{2}h(x+y)$. Draw freehand a figure to show what the letters represent. Make (i) h, (ii) x the subject.

22. Fig. 142 shows a symmetrical cross ; its area is A sq. in. Express A in terms of b, c and then make c the subject. What is c if $A = 108$, $b = 3$?

23. The sum of the angles of an n-sided polygon is r right angles. (i) What is r in terms of n ? (ii) Express n in terms of r.

24. Find a formula for the number n which is less than the number N by R per cent. Then make N the subject.

25. Fig. 126, p. 119, represents a semicircular disc of perimeter p in. ; express p in terms of π, d, and then make d the subject taking $\pi = \frac{22}{7}$.

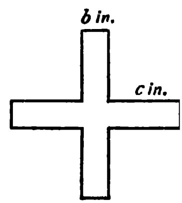


FIG. 142.

26. The Treasury formula for rating the power of motors is $H = \frac{2}{5}nd^2$. Make (i) n, (ii) d the subject of the formula.

27. From the formula for the velocity-ratio of a differential pulley system (Ex. VIII. b, No. 11), $n = \frac{2R}{R-r}$, obtain a formula with (i) r, (ii) R as subject.

28. If, with the data of No. 27, a load of W lb. requires an effort of P lb., the efficiency E is given by the formula, $E = \frac{(R-r)W}{2RP}$. Make R the subject.

[*For a revision exercise on Ch. VI-VIII, see Appendix, Ex. R. 4, p. 263.*]

TEST PAPERS, A. 26-35

A. 26

1. (i) Add together $2a^3-4a-3$, $5+3a-4a^2$, $3(a^3+a^2-1)$.
   (ii) Multiply $9c^2d^2$ by $3c^3d^3$ and divide the result by cd^5.

2. A train travels v miles per hour northwards; t hours after passing A, it is x miles north of A. What is x?

Evaluate x and interpret the results, (i) if $v=40$, $t=\frac{1}{2}$; (ii) if $v=-40$, $t=\frac{1}{4}$; (iii) if $v=40$, $t=-\frac{1}{8}$; (iv) if $v=-40$, $t=-\frac{1}{10}$.

3. (i) Solve the equation, $5(4x+5)=5(x+2\frac{1}{2})-\frac{1}{2}(7x+12)$.
   (ii) What can you say about z if $3(z^2+1)=30$?

4. In Fig. 143, ABCD and APQR are squares. $\mathsf{AB}=(n+1)$ inches, $\mathsf{AP}=(n-1)$ inches; what are the lengths of PB and RD?

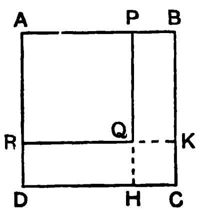


FIG. 143.

What is the area of the rectangle QRDH? Use the figure to simplify $(n+1)^2-(n-1)^2$.

5. Use the Centigrade-Fahrenheit formula, $\mathsf{C}=\frac{5}{9}(\mathsf{F}-32)$ to find k if $k°$ F. is the same temperature as $(-3k°)$ C.

A. 27

1. (i) Add together $\frac{1}{2}(2a+b-c)$, $\frac{1}{2}(2b+c-a)$, $\frac{1}{2}(2c+a-b)$.
   (ii) Simplify $3c^2d^4\times 2c^3d^2$.

2. (i) If $a=x^2+y^2$, $b=x^2-xy$, find the value of ab when $x=2$, $y=-1$.
   (ii) If $(2n)$ km. $=(n+\frac{1}{4}n)$ miles, express y miles in km.

3. (i) Solve the equation, $x(x+5)-2x(3-x)=3(x^2+1)$.
   (ii) For what value of W is $\frac{1}{2}(\mathsf{W}+1\frac{1}{2})$ equal to $\frac{1}{3}(\mathsf{W}+2\frac{1}{2})$?

4. Simplify (i) $\frac{-8x^2}{-2x}$; (ii) $\frac{-8x^2}{(-2x)^2}$; (iii) $-\frac{(-8x)}{(-x)}$.

5. 4 tickets, at $(p+3)$ shillings each, cost the same as $(p-3)$ tickets, at 8 shillings each. What is the cost of $(p+3)$ tickets at $(p-3)$ shillings each ?

A. 28

1. Simplify (i) $\frac{a}{2}+\frac{a}{3}+\frac{a}{6}$; (ii) $r^2s^3 \div \frac{r}{s^2}$; (iii) $1-\frac{s-t}{s}$

2. In Fig. 144, AB = AC and BD = BC ; find x in terms of y.

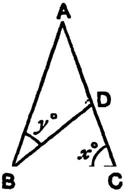


FIG. 144.

3. (i) Solve the equation $\frac{1}{5}(7a+5)-12=\frac{1}{7}(2a+1)$.

(ii) For what value of c is $x=2$ a root of the equation,

$$c(x^2+6)=x(c+6)\ ?$$

4. (i) Simplify $\frac{6p^3}{-4p^2q}$;

(ii) From $a\left(a+\frac{1}{a}\right)$ take $b\left(b+\frac{1}{b}\right)$.

5. A boy buys 10 dozen papers at 8d. a dozen ; he sells some at 1d. each and returns the rest, for which he gets 7d. a dozen. He gains half-a-crown. How many does he sell ?

A. 29

1. (i) Multiply $2x-3y+z$ by 5 and subtract the result from 3 times $(z-y-x)$.

(ii) Simplify $\frac{r^3}{s} \div \frac{s^3}{r}$.

2. Find in terms of c the perimeter of a rectangle, l inches long, b inches broad, if $\frac{l}{3}=\frac{b}{2}=\frac{c}{5}$.

What is the area of the rectangle, in terms of c ?

3. (i) Solve the equation, $\frac{1}{2}(x+18)-\frac{1}{3}(3x-4)=5(6-x)$.

(ii) What can you say about y if $(y+1)^2$ equals 49 ?

4. Copy and fill in the blanks in

(i) $4R^2-8Rr=4R(\quad)=-4R(\quad)$;

(ii) $a-b-c-d=a-(\quad)\quad d$.

5. If in Fig. 144, with the data of A. 28, No. 2, $y=\frac{1}{2}x$, find $\angle BAC$.

A. 30

1. If $a=3$, $b=-2$, $c=0$, $d=1$, find the values of

$$\text{(i) } a^2+b^2+2cd\,; \quad \text{(ii) } \frac{a-b}{c-d}\,; \quad \text{(iii) } a\div(b-c)+d.$$

2. I take t minutes to go from my house to the station, when I walk at $3\frac{3}{4}$ miles per hour. How long do I take if I run at $6\frac{1}{4}$ miles per hour ?

3. (i) Solve the equation, $\frac{2-n}{2}=\frac{2+n}{3}-\frac{8+n}{4}$.

(ii) Find values of a, b for which the expressions $\frac{a}{2}$, $\frac{a}{3}+\frac{b}{4}$, $\frac{a}{6}+\frac{1}{3}$ are equal.

4. From $(4x^2-6xy)\div2x$ take $(6xy-4y^2)\div2y$.

5. In an exercise, containing 36 questions, a boy is told to do each question numbered $(5n-3)$, where n is an integer. How many questions is this ?

Another boy is told to do questions 1, 5, 9, 13, 17, ... in the exercise. What formula would give this selection of questions ?

A. 31

1. (i) Simplify $\frac{2x-1}{4}-\frac{1+x}{6}$; divide the result by $\frac{1}{3}$.

(ii) Multiply $4pq$ by $4qr$ and divide the result by $4pr$.

2. If water flows at v ft. per sec. through a pipe of diameter d inches, the pipe delivers g gallons per minute where $g=2vd^2$. Use this formula to find the velocity of water in a pipe of diameter $\frac{5}{8}$ inch, which fills a tank holding 50 gallons in 12 minutes.

3. Solve the equations, (i) $d+\frac{1}{2}\pi d=1$, where $\pi=\frac{22}{7}$;

$$\text{(ii) } \frac{5x-12}{4}-\frac{7x+10}{11}=1.$$

4. If $P = a^2 - 3ab$ and $Q = ab - 2b^2$, express $P - 2Q$ in terms of a, b. Also find $\frac{P}{Q}$ if $a = -1$, $b = 2$.

5. At an examination, one-quarter of the candidates fail. At the next examination, there are 84 more candidates and 8 more failures ; on this occasion, one-fifth of the candidates fail. How many candidates were there at the first examination ?

A. 32

1. Simplify (i) $\left(\frac{1}{a} - \frac{a-1}{a^2}\right) \div a$; (ii) $\frac{2}{3}r^3s \times 3\frac{3}{4}rs^2$.

2. A graph is to be drawn showing the connection between the number of passengers in a train during a journey from London to Bournemouth and the distance from London. Which quantity should be measured along the axis across the page ?

If the number of passengers varies from 117 to 206, and if the length of axis is 8 inches, show how you would graduate it.

3. (i) Solve the equation, $\frac{x+5}{15} - \frac{x-5}{10} = 1 + \frac{2x}{15}$.

(ii) If $L = l(1 + kt)$, find t when $L = 1\frac{1}{4}l$ and $k = \frac{1}{50}$.

4. Simplify (i) $\frac{p+q}{2q} + \frac{q-p}{3q} - 1$; (ii) $\left(\frac{1}{2n}\right)^2(-2n)^2$.

5. I walk at 4 m.p.h. and run at 6 m.p.h. I find that I can save $3\frac{3}{4}$ minutes by running, instead of walking, from my house to the station. How far off is the station ?

A. 33

1. (i) Add together $\frac{3r+s}{6}$, $\frac{r-2s}{9}$, $\frac{s-r}{12}$.

(ii) Simplify $12a^6b^4c^2 \div 6a^3b^2c$.

2. To stay n days at a hotel costs $13n$ shillings if $n > 4$ and costs $14n$ shillings if $n < 5$, where n is an integer. Interpret this statement in words.

3. (i) Solve the equation, $\frac{1}{2}(x + 101) - \frac{2}{3}(x + 99) = 0$.

(ii) If $uv = 80$ and if $u = 5v$, what can you say about the value of u ?

4. If $2s = a + b + c$, express $s - a$ in terms of a, b, c. Also show that $\{(s-a) + (s-b) + (s-c)\}$ is equal to s.

5. The volume of metal in a tube of length l in., external radius R in., made of metal t in. thick is $\pi tl(2R - t)$ cu. inches. Find the internal radius of a tube 8 in. long, made of metal $\frac{1}{4}$ in. thick, if the volume is 33 cu. inches. Take $\pi = \frac{22}{7}$.

A. 34

1. Find the value of $a+b$ if $\frac{1}{2}=\frac{1}{3}+\frac{1}{a}$ and $\frac{1}{3}=\frac{1}{4}+\frac{1}{b}$.

2. If the base of a segment of a circle is $2k$ cm., and if its height

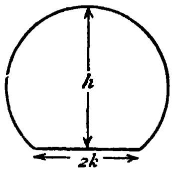


FIG. 145.

is h cm., then the radius of the circle is r cm., where $2rh=h^2+k^2$. What is the radius, if the height is 4·5 cm. and the base is 3 cm.?

3. (i) Solve the equation, $\frac{x-6}{13}-\frac{3-2x}{14}=3\frac{1}{2}$.

   (ii) If $\left(\frac{4n}{3}+1\right)$ is three times as large as $\left(\frac{3n}{5}-2\right)$, prove that $2n$ is five times as large as $\left(\frac{n}{3}+1\right)$.

4. Simplify (i) $3(5p-2q)-2(4p+3q)$;

   (ii) $\frac{8c^2x^3y}{6a^2b^3c^3}\div\frac{\frac{1}{2}xy}{15abc^2}$.

5. In Fig. 146, if AB = AC and BC = BD, find z in terms of y.

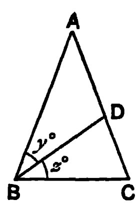


FIG. 146.

Also find y if $z=y+35$.

A. 35

1. Simplify (i) $\frac{1-b}{1+b}+\frac{2b}{b+1}$; (ii) $\left(c+\frac{1}{c}\right)\div\frac{1}{c}$;

   (iii) $\frac{t}{0\cdot6}-\frac{2t}{1\cdot5}$.

2. If $R = 3x - y$ and $r = 2y - x$ and $2R + r = 7\frac{1}{2}$, find the value of x. Also, if $R = 3r$, find the value of y.

3. (i) Solve the equation, $\dfrac{2x-9}{28} - \dfrac{x-2}{21} + \dfrac{3}{28} = 0.$

(ii) If $h = \dfrac{r(1-n)}{1+n}$ and if $n = \frac{3}{7}$, find r in terms of h.

4. Copy and complete the following :

(i) $\dfrac{1}{s-r} = \dfrac{-1}{}$: (ii) $\dfrac{a-x}{a+x} = -\dfrac{}{x+a}$

5. A bookshelf will just hold $3x$ books, each $1 \cdot 6$ inches thick, or $2(x+1)$ books, each $2\frac{1}{4}$ inches thick. What is the length of the shelf ?

[*For additional test papers on Ch. I-VIII, see Appendix,* ***P.*** 6-15, *p.* 314.]

EXTRA PRACTICE EXERCISES FOR PART I

EXERCISE E.P. 1

Substitution (Chapter II)

If $a=3$, $b=5$, find the values of :

1. ab.	2. $a+b$.	3. $2a-b$.	4. ba^2.
5. $2a^2$.	6. $\frac{2a}{b}$.	7. $b-a$.	8. $1+ba$.

If $u=3$, $s=1$, $t=0$, find the values of :

9. us.	10. us^2.	11. uts.	12. $u^2+s^2+t^2$.
13. $2u-3s$.	14. $us+st$.	15. $3t$.	16. $4s^3-3ut$.

If $N=4$, $n=\frac{1}{2}$, find the values of :

17. Nn.	18. $\frac{1}{n}$.	19. Nn^2.	20. $N-5n$.
21. $\frac{N}{n}$.	22. $\frac{n}{N}$.	23. $2n^2$.	24. $\frac{1}{2}N^2$.

If $e=3$, $f=5$, $g=0$, find the values of :

25. $4ef$.	26. $2f^2$.	27. $3fg+ge$.	28. $5e^2-6f$.
29. $3e^2-f^2$.	30. fge^2.	31. e^3+2g^2.	32. $\frac{g}{ef}$.
33. $2(e+f)$.	34. $e(f+2)$.	35. $3(f-e)$.	36. $2e(f+g)$.

If $x=6$, $y=0$, $z=3$, find the values of :

37. $x+y+z$.	38. $2x-3y-2z$.	39. $xy+yz$.
40. $\frac{x+3}{z}$.	41. $\frac{1}{z}-\frac{1}{x}$.	42. $x+xz^2$.
43. $x(z-2y)$.	44. $\frac{yz}{x}$.	45. $x(x-2z)$.

If $m=1$, $n=2$, $t=3$, find the values of :

46. m^3.	47. mnt.	48. $m+n-t$.	49. n^2-mt.
50. $m+n$	51. n^3.	52. $t^2-m^2-n^2$.	53. $\frac{nt}{m}$.

If $c=\frac{2}{3}$, $d=\frac{1}{4}$, find the values of :

54. $3c-4d$.	55. $\frac{c}{d}$.	56. $\frac{1}{c}+\frac{1}{d}$.	57. dc^2.
58. $1-c$.	59. $6cd$.	60. $3c^2-4d^2$.	61. $\frac{1}{c^2}-d$.

If $p=4$, $q=9$, $r=1$, find the values of :

62. $p(q+r)$. 63. $(q-p)(p-r)$. 64. $p(5r-4)$.

65. $\frac{q+3r}{p}$. 66. $\frac{q}{p-1}$. 67. $\frac{q+r}{p+1}$.

68. $\sqrt{p}$. 69. $\sqrt{q}$. 70. $\sqrt{(pq)}$.

71. If $a=b^2-b$ and $b=5$, what is (i) a, (ii) $\frac{a}{b}$?

72. If $x=2y$ and $x=10$, what is (i) y, (ii) xy ?

73. If $2m^2+n^2=p$ and if $m=0$ and $n=1$, what is p ?

74. If $y=2x^2-3x+1$, what is y if (i) $x=1$, (ii) $x=2$, (iii) $x=3$?

75. If $y=(x+1)(x+3)$, what is y if (i) $x=0$, (ii) $x=1$, (iii) $x=2$?

76. If $x=2y$ and $y=3z$ and $z=4$, what is x ?

77. If $u=2\frac{1}{2}$ and $v=1\frac{2}{3}$, what is $\frac{1}{u}+\frac{1}{v}$?

78. If $l=\frac{2}{b}$, and $b=\frac{3}{4}$, what is (i) l, (ii) lb, (iii) $\frac{l}{b}$?

79. If $\frac{a}{b}=6$, what is a if $b=2$? What is b if $a=24$?

80. If $pq=56$ and if $q=4$, what is $\frac{p}{q}$?

81. If $\frac{1}{n}=1\frac{1}{2}$, what is n ?

EXERCISE E.P. 2

Generalisation (Chapter II)

Give general statements which include the following :

1. $\frac{3}{3}=1$; $\frac{6}{6}=1$; $\frac{11}{11}=1$; $\frac{x}{x}=\quad$.
2. $0\times 7=0$; $0\times 4=0$; $0\times 10=0$.
3. $1+\frac{2}{3}=\frac{3+2}{3}$; $1+\frac{7}{11}=\frac{11+7}{11}$; $1+\frac{5}{14}=\frac{14+5}{14}$.
4. $1\div\frac{2}{3}=\frac{3}{2}$; $1\div\frac{7}{11}=\frac{11}{7}$; $1\div\frac{5}{14}=\frac{14}{5}$.
5. $\frac{3\times 7}{7}=3$; $\frac{8\times 5}{5}=8$; $\frac{10\times 12}{12}=10$.
6. $2+4+6=4\times 3$; $7+9+11=9\times 3$; $12+14+16=14\times 3$.
7. $\frac{3\times 5}{3\times 7}=\frac{5}{7}$; $\frac{6\times 4}{6\times 11}=\frac{4}{11}$; $\frac{10\times 3}{10\times 8}=\frac{3}{8}$.
8. £3 $=3\times 20$s. ; £7 $=7\times 20$s. ; £12 $=12\times 20$s.
9. In £2, there are 2×8 half-crowns ; in £P, ...
10. In 7 feet there are 7×12 inches ; in s feet, ...
11. 100 minutes $=\frac{100}{60}$ hours ; 45 minutes $=\frac{45}{60}$ hours.

12. The cost of 7 lb. of sugar at 3d. per lb. is 3×7 pence.

13. If a boy scores 52 runs in 5 completed innings, his average is $\frac{52}{5}$ runs.

14. If a man earns £500 a year and spends £40 a month, he saves £$(500 - 40 \times 12)$ a year.

15. If a boy bicycles at 10 miles an hour, in $1\frac{1}{2}$ hours he goes $10 \times 1\frac{1}{2}$ miles ; in t hours he goes ....

16. If a boy bicycles at 8 miles an hour, a journey of 16 miles takes $\frac{16}{8}$ hours.

17. If a coal scuttle when empty weighs 3 lb. and when full weighs 14 lb., the coal in it weighs $(14 - 3)$ lb.

18. From 9 a.m. to 4 p.m., there are $[(12 - 9) + 4]$ hours.

19. If a clock loses 5 seconds each hour, it loses 2 minutes each day.

20. If a soldier's stride is 30 inches, he takes, in every hundred yards, $\frac{100 \times 36}{30}$ paces.

21. If £7 is shared by 10 people, each gets 14 shillings.
If £11 is shared by 10 people, each gets 22 shillings.

22. If the diameter of a circle is 6 cm., its radius is 3 cm.

23. £3. 11s. equals $12\{3 \times 20 + 11\}$ pence.

24. 15 miles per hour is the same speed as 22 feet per second.

25. The number midway between 9 and 17 is $\frac{9+17}{2}$.

26. Since $15 - 4 = 11$, $\therefore$ $15 = 11 + 4$.
If $a - b = c$, then ....

27. $\frac{1}{2}(3^2 + 5^2) - 4^2 = 1$; $\frac{1}{2}(8^2 + 10^2) - 9^2 = 1$; $\frac{1}{2}(15^2 + 17^2) - 16^2 = 1$.

28. If a railway fare is 15 pence, 4 whole tickets and 3 half tickets cost $15(4 + \frac{3}{2})$ pence.

29. The square of 14 is four times the square of 7.
The square of 22 is four times the square of 11.

30. Below 10, there are 5 odd numbers. Below 18, there are 9 odd numbers. Below 46, there are 23 odd numbers.

EXERCISE E.P. 3

Simple Equations (Chapter III)

Solve the following equations and check each answer :

1. $4t = 32$.	2. $R + 4 = 32$.	3. $a - 4 = 32$.
4. $\dfrac{b}{4} = 32$.	5. $2p - 14 = 0$.	6. $14 = 3c$.
7. $y + \frac{1}{2} = 2$.	8. $2z - 1 = 0$.	9. $10 - z = 5$.

10. $\frac{2n}{3}=12$.
11. $\frac{y}{5}=5$.
12. $2\frac{1}{2}=d-\frac{1}{2}$.
13. $7-h=4$.
14. $0=2k-1$.
15. $\frac{t}{2}=1\frac{2}{3}$.
16. $3-\frac{1}{2}p=0$.
17. $x-2\frac{1}{2}=3\frac{1}{2}$.
18. $z+1{\cdot}4=3{\cdot}1$.
19. $4{\cdot}5-l=2{\cdot}5$.
20. $3r=1$.
21. $3s-1=1$.
22. $5-5p=5$.
23. $\frac{2h}{3}=\frac{2}{3}$.
24. $0{\cdot}4k=4$.
25. $\frac{a}{4}=\frac{1}{4}$.
26. $12-\frac{1}{2}b=7$.
27. $0=3-9c$.
28. $1{\cdot}5n=6$.
29. $0{\cdot}7p=2{\cdot}8$.
30. $0{\cdot}1s=0$.
31. $2r+5=11-r$.
32. $3t-7=2t-3$.
33. $x-1=5x-9$.
34. $2(y-1)=14$.
35. $3(z+2)=27$.
36. $0=4v-9$.
37. $10a=5a$.
38. $10b=5$.
39. $c-\frac{c}{2}=10$.
40. $5(n-1)=7$.
41. $p+\frac{3}{4}p=28$.
42. $\frac{r}{4}=1+\frac{r}{5}$.
43. $\frac{s}{3}-\frac{3}{4}=0$.
44. $\frac{1}{2}(t+1)=5$.
45. $0=\frac{1}{3}(c-10)$.
46. $\frac{3y}{4}-\frac{y}{2}=1$.
47. $\frac{z}{2}+\frac{z}{4}=0$.
48. $\frac{5x}{2}=1{\cdot}6$.
49. $3-\frac{1}{2}b=1+b$.
50. $\frac{3c}{4}-2=2\frac{1}{2}$.
51. $\frac{d-2}{3}=4$.
52. $\frac{5}{n}=\frac{2}{3}$.
53. $2\frac{1}{2}=\frac{2}{p}$.
54. $\frac{1}{2l}=\frac{5}{6}$
55. $1-\frac{3x}{4}=x-6$.
56. $\frac{4y}{7}-\frac{y}{2}=\frac{1}{3}$.
57. $\frac{z}{2}-\frac{z}{3}=\frac{z}{4}$.
58. $l-0{\cdot}4l=30$.
59. $\frac{1}{d}+\frac{1}{2d}=\frac{1}{3}$.
60. $\frac{r}{1{\cdot}5}=1{\cdot}2$.

EXERCISE E.P. 4

Like and Unlike Terms (Chapter IV)

Simplify, where possible, the following expressions. *If there is no shorter form, say so.*

1. $u+2u$.
2. $6c-2c$.
3. $3a+3a$.
4. $3b+2+b$.
5. $pq+qp$.
6. $10t^2-5t^2$.
7. $3b+0$.
8. $3m+2m$.
9. $5d^2-5$.
10. $4e^2-4e^2$.
11. $2k+\frac{k}{2}$.
12. $r-\frac{1}{2}r$.
13. $2s+2$.
14. $\frac{t}{2}+\frac{3t}{2}$.
15. $n+\frac{n}{3}$.

16. $2z + z + 3z$. 17. $p^2 + 4p^2 - 5p^2$. 18. $3bc + 2c$.
19. $rs + sr$. 20. $7R - 5r$. 21. $2a^2b + ba^2$.
22. $4c^2 - 4c$. 23. $8st - 4ts$. 24. $nt - \frac{1}{4}nt$.
25. $6A + 3A - 9A$. 26. $10m^3 - m^3$. 27. $p + pq$.
28. $\frac{1}{3}y + \frac{5y}{12}$. 29. $2\frac{1}{2}d - \frac{u}{6}$. 30. $4 \times 5l - 5l$.
31. $2x + 3y - x + y$. 32. $4 + 4c + 2g - 2$.
33. $p + q + q + p + 1$. 34. $f^2 - 3fg - 5fg + f^2$.
35. $r^2s + rs^2 + sr^2$. 36. $2u - v + u + 3v + 3$.
37. $3cd + 2dc + c + d$. 38. $ab - ac + bc - ba + ca - cb$.
39. $t^2 - 2t + 4 + 2t^2 - 3t - 1$. 40. $4yz - 3xz + zy - xy$.
41. $x^3 - 5x + 4 - x^3 + 6x - 3$. 42. $p^2 + q^2 - p^2 + q^2$.
43. $c^2 - 1 - c - 1$. 44. $2 + 3e^2 + e^4 - e^2 + 1$.
45. $aba + bab$. 46. $4 - 6R + 2 - 3R$.
47. $10u + 10u^3 - 10u^2 - 10$. 48. $ccdd + cdcd + dccd$.

EXERCISE E.P. 5

Products and Quotients (Chapter IV)

Simplify the following :

1. $x^4 \times x^4$. 2. $b^6 \div b^2$. 3. $3c \times 3c$. 4. $4d \div 5e$.
5. $p^3 \times p^3$. 6. $2t \times t^2$. 7. $3rs \div 2s$. 8. $6y^6 \div 2y^2$.
9. $e \times e^4$. 10. $6a^2b \div 2a$. 11. $c^4 \div c^4$. 12. $5d \times 5d$.
13. $10p \div p^2$. 14. $4t^4 \times 2t^2$. 15. $8r^2s^2 \div 2rs^2$. 16. $4a^3 \times 2a$.
17. $2x^2 \times 3xy^2$. 18. $8p^4 \div 2p$. 19. $4k \times \frac{1}{2}k$. 20. $\frac{l}{2} \times \frac{l}{3}$.
21. $5n \div \frac{1}{2}$. 22. $pq \times pr$. 23. $6s \div 8s$. 24. $\frac{n}{2} \times \frac{2}{n}$.
25. $3ab^2 \times 3a^2b$. 26. $3r^2s \div 2rs$. 27. $6z^6 \div 3$. 28. $12 \div \frac{3}{c}$.
29. $t \times 2t \times 3t$. 30. $2p \times 3q \times pq$. 31. $2x^2 \times 3xy \times 4y^2$.
32. $a^3 \times a^3 \div a^3$. 33. $b^2 \times b^4 \div b^6$. 34. $12c^6d^6 \div 3c^3d^2$.
35. $2ab \times 2bc \times 2ca$. 36. $4rs \times 4rt \div 4st$. 37. $\frac{a^2}{2} \times \frac{b^2}{3} \div \frac{ab}{5}$.
38. The square of $4bc^3$. 39. A square root of $4d^8$.
40. The cube of $3n^2$. 41. The square of $3r^3$.
42. A square root of $16x^2y^4$. 43. The cube of $2y^3z^3$.
44. The cube root of $8s^6$. 45. The square of $5pq^2r^3$.

46. $(2a)^2+(3a)^2$.
47. $(4b)^3\div(2b)^2$.
48. $(3c)^2\times 3c^2$
49. $(p^4)^2\div p^2$.
50. $(2pq)^2\times 3q$.
51. $(xy^2)^2\times 2x^3$.
52. $(3t)^3-(2t)^3$.
53. $(2v)^2\times(3v)^3\div 6v^2$.
54. $(4z^2)^2\div 4z$.
55. $a^2\times(2ab)^2$.
56. $(2p^2)^3\div(2p^3)^2$.
57. $4a^3b^8c^3\div 4abc$.
58. $n\times(2n)^2\times(3n)^3$.
59. $(3ab^3)^2\div 3ab$.
60. $t(2t)^3\div t^2$.

EXERCISE E.P. 6

H.C.F. and L.C.M. (Chapter IV)

Find the following :

1. H.C.F. of $6r$, $3rs$.
2. H.C.F. of a^2b, ab^2.
3. L.C.M. of $2pq$, $6p^2q$.
4. L.C.M. of $3y^2$, $2yz$.
5. H.C.F. of $5a^2b$, $10ab^3$.
6. H.C.F. of $6c^3x^4$, $9c^2x^3$.
7. L.C.M. of $2xy$, $2xyz$.
8. L.C.M. of $4\,abxy$, $14byz$.

Find the H.C.F. and L.C.M. of the following :

9. $3x^2$, $2x^3$.
10. $5a^2b$, $10b^2$.
11. $6y^3$, $8y^2z$.
12. $4x^3$, $6y^3$.
13. $4pq$, $5r^2$.
14. $8yx$, $10xy$.
15. 12, $10t$.
16. y^6z^2, y^3z.
17. $9ef^2$, $6fg^2$.
18. x^2y, xy^2.
19. $4b^3$, $9c^3$.
20. $12b^2x$, $8aby$.
21. $10x^2$, $12xy$, $2x^3$.
22. $6a^3b^3$, $9a^2bc$, $12a^4b^2c^2$.
23. $2z^2$, $3z^3$, $4z^4$.
24. $4r^2$, $5rs$, $6s^2$.
25. $6x$, $2xy$, $2y^3$, $4x^2$.
26. $2x^2yz$, $3xy^2z$, $4xyz^3$.
27. 8, $10p$, $4p^3$, $3q^2$.
28. $9x^3y$, $6y^3z$, $12x^2y^2z^2$.
29. $10x^3$, $15abx^2$, $20cxy$.
30. $(2ax)^2$, $(6bx)^2$, $(2cx)^2$.

EXERCISE E.P. 7

Fractions (Chapter IV)

Simplify, where possible, the following :

1. $\frac{a^2}{a}$.
2. $\frac{b^3}{b^2}$.
3. $\frac{3c}{3}$.
4. $\frac{4d}{d}$.
5. $\frac{5e}{5e}$.
6. $\frac{a^2}{b^2}$.
7. $\frac{2a}{2b}$.
8. $\frac{p^3}{3p}$.
9. $\frac{r^6}{r^2}$.
10. $\frac{s}{st}$.
11. $\frac{u}{u^2}$.
12. $\frac{t+2}{2}$.
13. $\frac{v^3}{v^6}$.
14. $\frac{a+2}{b+2}$.
15. $\frac{2a}{2b}$.
16. $\frac{2ac}{2ad}$.
17. $\frac{2xy}{x^2}$.
18. $\frac{z}{z}$.
19. $\frac{a}{ab}$.
20. $\frac{c^2}{2cd}$.

21. $\frac{2e}{2e^2}$. 22. $\frac{3rs}{3sr}$. 23. $\frac{4g^4}{h^4}$. 24. $\frac{kl}{k^3l^3}$. 25. $\frac{4m^4n^4}{4mn}$.

26. $\frac{p^3}{3qr}$. 27. $\frac{6t^2u^2}{9u^2v^2}$. 28. $\frac{w^2}{w^2x^2}$. 29. $\frac{yz}{z}$. 30. $\frac{a^6b^6}{a^2b^2}$.

31. $\frac{c^6}{3c^2}$. 32. $\frac{x^3}{(2x)^2}$. 33. $\left(\frac{3d}{2d}\right)^2$. 34. $\frac{x^2}{y^2z^2}$. 35. $\frac{\;}{(pqr)^2}$.

36. $\frac{a}{6}+\frac{a}{8}$. 37. $\frac{2b}{3}-\frac{b}{2}$. 38. $\frac{7a}{3}+\frac{13a}{6}$.

39. $1-\frac{c}{3}$. 40. $\frac{3d}{2}+1$. 41. $\frac{e^2}{2e}-\frac{e^3}{6e^2}$.

42. $\frac{1}{t}+\frac{1}{2t}$. 43. $\frac{1}{r}-\frac{1}{3r}$. 44. $\frac{c}{cd}-\frac{1}{d}$.

45. $n-\frac{1}{n}$. 46. $1+\frac{1}{s}$. 47. $gh+\frac{g}{h}$.

48. $\frac{2}{l}-\frac{l}{2}$. 49. $\frac{1}{2mn}+\frac{3}{n^2}$. 50. $\frac{p}{q}-\frac{p^2}{q^2}$.

51. $r-\frac{s}{t}$. 52. $\frac{1}{u}-\frac{v}{ur}$. 53. $\frac{1}{6yz}+\frac{1}{4z^2}$.

54. $\frac{a}{10bc}-\frac{b}{15ac}$. 55. $r\div\frac{r}{s}$. 56. $\frac{1}{b}\div\frac{1}{d}$.

57. $ef\div\frac{1}{e}$. 58. $1\times\frac{p}{q}$. 59. $\frac{x}{y}\div 1$.

60. $\frac{a^2}{b^2}\div\frac{b}{a}$. 61. $\frac{cd}{ce}\div de$. 62. $b^6\div\frac{1}{b^2}$.

63. $\left(\frac{2c}{d}\right)^2\div cd$. 64. $t\div\frac{1}{t}$. 65. $\frac{z}{3yz}+\frac{x}{6xy}$.

EXERCISE E.P. 8

Brackets (Chapter VI)

Simplify the following :

1. $a+(b-3c)$. 2. $d+(b-d)$. 3. $2f-(f+g)$.
4. $(k+l)-(k-l)$. 5. $p+2q-(3q-p)$. 6. $(r-2s)-s$.
7. $3t+(t-2s)$. 8. $4y-(x-3y)$. 9. $(x-y)+(y-x)$.
10. $b^2-(b^2-c^2)$. 11. $(a-2z)-(z-2a)$. 12. $(1+n)-(1-n)$.
13. $(c-d)+(d-e)$. 14. $p-(p-q)-q$. 15. $1+(1-x)-(2-2x)$.

Solve the equations :

16. $3-(1-t)=2t$. 17. $(3x-4)-(5+x)=0$.
18. $4n=10-(2n+1)$. 19. $5-(2p-1)+(4-p)=0$.

Fill in the blanks in the following :

20. $a-c+e=a-(\ldots\ldots)$.

21. $p-q-r=p-(\ldots\ldots)$.

22. $x-y+z=z-(\ldots\ldots)$.

23. $r+s+t=3r-(\ldots\ldots)$.

24. From $2a-b+c$ subtract $a+b+c$.

25. From $3p+2q-3r$ subtract $r-2q$.

26. By how much does $y-x+z$ exceed $x-z-y$?

27. Subtract $r-s$ from the sum of $r+2s-t$ and $r-s+2t$.

EXERCISE E.P. 9

Brackets (Chapter VI)

Simplify the following :

1. $2(c-d)-(c+d)$.

2. $3(f+h)-2(h-f)$.

3. $5-2(1-k)$.

4. $p-3(r-p)$.

5. $2q+5(a+2q)$.

6. $4(u-v)-3(u+v)$.

7. $x(x+y)-x(x-y)$.

8. $x(y+x)-y(y+x)$.

9. $yz-z(y-z)$.

10. $(x^2-5x)\div x$.

11. $\dfrac{t-1}{3}+\dfrac{t+1}{2}$.

12. $\dfrac{p-2}{3}-\dfrac{p+1}{4}$.

13. $\frac{1}{2}(a+b)-\frac{1}{3}(a-b)$.

14. $\frac{3}{4}(c-d)-\frac{1}{4}(c+d)$.

15. $\frac{4}{5}(p+2q)-\frac{3}{5}(p-q)$.

16. $\dfrac{2r-3}{10}-\dfrac{2r+3}{15}$.

17. $c-\dfrac{c+d}{2}$.

18. $1-\dfrac{2x+3y}{5x}$.

19. $\dfrac{2s}{3t}-\dfrac{6-s}{12t}$.

20. $\dfrac{m-n}{12}-\dfrac{m+3n}{60}$.

21. $\dfrac{4a^2-6ab}{2a}$.

22. $c-c\left(1-\dfrac{1}{c}\right)$.

23. $\dfrac{e-2f}{3f}-\dfrac{e-2f}{4e}$.

24. $\dfrac{a^3+2b^3}{b^3}-\dfrac{a^4-2b^3}{a^3}$.

25. $3(x+y-2z)-3(x-y+2z)$.

26. $r(s-t)+s(t-r)+t(r-s)$.

27. $(6x^2y-8xy^2)\div 2xy$.

28. $p\left(1+\dfrac{1}{p}\right)-q\left(1+\dfrac{1}{q}\right)$.

29. $a-\{a-(1+a)\}$.

30. $2r-3\{4-\overline{r-1}\}$.

31. $3b+c-\{4b-3(b-c)\}$.

32. $2\{x-\overline{y-x}\}-3(x+y)$.

33. $y-2\{z-4(y-z)\}$.

34. $a\{a(a+b)-b(a-b)\}$.

Copy and complete the following :

35. $a-2b-2c=a-2(......)$. 36. $4x^2-6xy=2x(......)$.

37. $2a-2c-3x-3z=2(......)-3(......)$.

38. $a^2+ab-p^2+pq=a(......)-p(......)$.

39. $3l+m-n=4l-(......)=3(l+m)-(......)$.

40. $1+x+x^2+x^3=1+x+x^2(......)=1+x^2+x(......)$.

41. Add $x+2(y-z)$ to $x-2(y+z)$.

42. Subtract $a-(b-c)$ from $c-(b-a)$.

43. Multiply $x-(x-y)$ by $y-(y-x)$.

44. What must be added to $p-(q+r)$ to give $p+(q+r)$?

45. What must be subtracted from $2(r-s)+t$ to leave $2(r-t)$?

EXERCISE E.P. 10

Positive and Negative Numbers (Chapter VII)

1. Subtract :
(i) $-a$ from a ; (ii) b from $-b$; (iii) $5y$ from $7y$;
(iv) $-3x$ from $5x$; (v) $5p$ from $-2p$; (vi) $-6z$ from $-2z$.

2. Add :
(i) $-a$ and a ; (ii) $-2b$ and $-3b$; (iii) $-c$ and $5c$;
(iv) $7x$ and $-4x$; (v) 0 and $-2y$; (vi) $-4z$ and $-z$.

3. Multiply :
(i) $-a$ by -1 ; (ii) $2b$ by -2 ; (iii) $-3x$ by -4 ;
(iv) $-y$ by y ; (v) $-z$ by $-z$; (vi) -5 by $2c$.

4. Divide :
(i) $-4a$ by -1 ; (ii) -2 by -2 ; (iii) $6y$ by -2 ;
(iv) $-9x$ by 3 ; (v) $-x^2$ by $-x$; (vi) $6ab$ by $-3b$.

If $a=-1$, $b=-2$, $c=3$, $d=0$, $x=-4$, $y=1$, write down the values of the following, Nos. 5-57 :

5. ab.	6. ac.	7. bd.	8. ax.	9. dy.
10. bc.	11. ay.	12. bx.	13. cd.	14. xy.
15. $\frac{b}{a}$.	16. $\frac{c}{a}$.	17. $\frac{x}{y}$.	18. $\frac{y}{a}$.	19. $\frac{a}{y}$.
20. $\frac{d}{y}$.	21. $\frac{d}{a}$.	22. $\frac{x}{b}$.	23. $\frac{b}{y}$.	24. $\frac{b}{x}$.
25. b^2.	26. a^3.	27. x^2.	28. a^4.	29. d^2.
30. $a-b$.	31. $b+c$.	32. $d-a$.	33. $x-y$.	34. $c-x$.
35. $y+a$.	36. $x-b$.	37. $a-c$.	38. $b-d$.	39. $y+b$.
40. $y-a$.	41. $x-c$.	42. $2x+y$.	43. $3b-c$.	44. $d-2a$.

45. $3y-b$. **46.** $3b-2d$. **47.** $b-2a$. **48.** $a-2x$. **49.** $c+4a$.
50. $2c-6a$. **51.** $3y-c$. **52.** $2b-x$. **53.** $3x-4a$.
54. $a(b-c)$. **55.** $b(d-a)$. **56.** $x(y-b)$. **57.** $c(a-x)$.

Simplify the following :

58. $(-p)(3p)$. **59.** $(-2r)(-r)$. **60.** $(+3s)(-2t)$.
61. $(-8t)\div(-2)$. **62.** $(-6k)\div 3$. **63.** $(4l)\div(-1)$.
64. $-(-4a)$. **65.** $+(+3b)$. **66.** $-(+2c)$.
67. $(-3)(2x)$. **68.** $0\times(-3y)$. **69.** $0\div(-4z)$.
70. $3p-5p-p$. **71.** $a-4a+2a$. **72.** $-t-3t-4t$.
73. $x+(-3x)$. **74.** $(-y)+(-y)$. **75.** $2z-(-z)$.
76. $-(-a)+(+a)$. **77.** $-(+c)+(-c)$. **78.** $+(-p)-(-p)$.
79. $-(+2b)-(-2c)$. **80.** $3r-(-3s)$. **81.** $(-2k)+(-l)$.
82. $(-a)\div(ab)$. **83.** $c\div(-c^2)$. **84.** $(-d)\div(-2d)$.
85. $\left(-\frac{p}{q}\right)\div(-p)$. **86.** $\left(\frac{x}{y}\right)(-y)$. **87.** $(-x^2)\div(xy)$.
88. $\left(-\frac{2}{r}\right)\times\left(\frac{r}{-2}\right)$. **89.** $(-ab)\div(-2b)$. **90.** $\left(\frac{a}{-1}\right)\times b$.
91. $(-1)(b-a)+(-a)$. **92.** $c+(-2)(c-d)$.
93. $(x^2-2xy)\div(-x)$. **94.** $(-3)(p-q)+(-2)(q-p)$.
95. $(r-s)\div(-1)+s$. **96.** $-2y+(-3)(x-y)$.
97. $0+(-1)(z-y)$. **98.** $(a-b)-(b-a)$.
99. $(-2)(3x)+(1-1)(-4x)$. **100.** $(-2x)^2-2x^2$.

EXERCISE E.P. 11

Substitution and Brackets (Chapter VII)

If $a=2$, $b=-1$, $c=-3$, $d=0$, find the values of :

1. $a-(b+c)$. **2.** $b-(a-c)$. **3.** $c-(b-a)$.
4. $2b-3(d-c)$. **5.** $3c-2(a-d)$. **6.** $4d-(b-c)$.
7. $(a-b)(a-c)$. **8.** $(b-c)(d-a)$. **9.** $(c-b)(a+b)$.
10. $\frac{a+d}{b-c}$. **11.** $\frac{a(b-c)}{b(a+c)}$. **12.** $\frac{ad-bc}{ab-cd}$.
13. b^2-5b-3. **14.** b^2+bc+c^2. **15.** $3a^2+2ac-c^2$.
16. $3c^2-2cd-3d^2$. **17.** $2ab+3bc+4ca$. **18.** b^3-c^3+3abc.
19. $2a^2-c\{3b-2(a+c)\}-(a-b)^2$.
20. $a\{b-c[d-3(b-c)]+a\}-(a+b+c)^3$.
21. $c\{b-a[2(2a+c)-(4b+c)(a+d)]\}$.
22. $(a-b)(b-c)(c-a)-\{a-b[b-c(c-a)]\}$.

Simplify the following :

23. $3x - \overline{2x - 3y} + 3 \cdot \overline{y - x}$.

24. $4e - 2(3e - f) - 3(f - e)$.

25. $p - 2q - \{3q - 2 \cdot \overline{p - q}\}$.

26. $3(y - z) - \overline{z - y} - 2y$.

27. $3 \cdot \overline{r - 2s} - 2 \cdot \overline{s - 3r}$.

28. $2\{u - 3 \cdot \overline{u - 1}\} - u$.

29. $x - \{x - [2x - 3(x - y) - y] - y\} - (x + y)$.

30. $2(z - y) - 3\{y + 2[z - 3(y + z)] - z\}$.

EXERCISE E.P. 12

Miscellaneous Operations with Negative Numbers (Chapter VII)

1. Subtract $-a - b + c$ from a.
2. Divide $x^2 - 3xy - x$ by $-x$.
3. Simplify $\dfrac{x}{-y} + \dfrac{-x}{y} - \dfrac{-x}{-y}$.
4. Copy and fill in the blank space in $x - y = (-1)(\quad)$.
5. Simplify (i) $(-a)(-b)(-c)$; (ii) $(-x^3)^2$;
   (iii) $(-y)^2(-y^2)$; (iv) $(2p)(-3q) \div (-1)^3$.
6. Subtract $4x^2 + 7$ from $x^2 - 5x - 2$.
7. Add $-(1 - 2x)$ to $3(x + 1)$.
8. Divide $x^4y^2 - x^2y^4$ by $-xy$.
9. Multiply $1 - (2 - a)$ by $-a$.
10. Simplify $\dfrac{b^2}{-b} + \dfrac{b^2 - b^4}{-b^2} + (-b)^2$.
11. If $(x - a)(y - a) = c^2$, and if $a = -1$, $c = -1$, $y = 1$, find the value of x.
12. If $r + s + 4 = 0$ and $s = -1$, what is the value of $\dfrac{r}{s}$?
13. If $x = -\frac{1}{2}$ and $y = -\frac{1}{3}$, what is the value of $\dfrac{1}{x} - \dfrac{1}{y}$?
14. Simplify $(a - 2a^2) \div (-a) + (2b^2 - b) \div (-b)$.
15. If $pq = p + q$ and $q = -1$, what is p ?
16. Can you find two numerical values of x such that the square of $x - 7$ is 16 ?
17. Multiply $\dfrac{1}{-a} + \dfrac{1}{-b} - \dfrac{1}{-ab}$ by $-ab$.
18. Divide $x^3y^2z^2 - x^2y^3z^2 + x^2y^2z^3$ by $-xyz^2$.
19. Copy and fill in the blank space in $2r - 3s = (s - r) - (\quad)$.
20. If $y + z = 0$, simplify $\dfrac{y^2 + z^2}{yz}$.

EXERCISE E.P. 13

Simple Equations (Chapter VIII)

Solve the following equations :

1. $\frac{3}{4}t = 7\frac{1}{2}$.

2. $5 + \frac{2v}{3} = 0$.

3. $\frac{2}{3}(p - 1) = 1$.

4. $\frac{1}{4}(r + 1) = \frac{1}{7}(r - 1)$.

5. $3x + 1{\cdot}25x = 3{\cdot}4$.

6. $\frac{y}{2} + \frac{y}{3} = y - 7$.

7. $3(n - 2) + 4 - 4(3 - n) = 0$.

8. $1\frac{2}{3}(c + 1) = c + 5\frac{2}{3}$.

9. $\frac{2y + 4}{3} = \frac{1}{4}y - 2$.

10. $\frac{t + 3}{5} = 8 - \frac{t - 1}{4}$.

11. $\frac{1}{r} + \frac{1}{2r} - \frac{1}{3r} = 2\frac{1}{3}$.

12. $\frac{s}{2} - \frac{s}{3} = \frac{s}{4} - 1$.

13. $\frac{2x + 7}{4} - \frac{x + 1}{3} = \frac{3}{4}$.

14. $\frac{a}{2} - \frac{5a + 4}{3} = \frac{4a - 9}{3}$.

15. $\frac{t + 1}{2} - \frac{t - 7}{5} = \frac{t + 4}{3}$.

16. $4\left(\frac{1}{n} - 3\right) = \frac{1}{n} - 2$.

17. $\frac{5}{6}(p - 2) = \frac{3}{4}(2p - 5)$.

18. $\frac{2z + 1}{4} = 3 - \frac{z - 2}{6}$.

19. $\frac{3}{2a} - \frac{2}{3a} = 1{\cdot}2$.

20. $\frac{3x + 2}{5} - \frac{2x + 5}{3} = x + 3$.

21. $\frac{2}{3}(y + 5) = y - \frac{1}{4}(y - 10)$.

22. $\frac{3(k - 3)}{4} - \frac{7k - 2}{5} = 2 - k$.

23. $\frac{x + 1}{2} + \frac{x + 2}{3} + \frac{x + 3}{4} + \frac{1}{4} = 0$.

24. $\frac{9x + 7}{2} - \left(x - \frac{x - 2}{7}\right) = 36$.

25. $t - \frac{t - 2}{3} = \frac{t + 3}{4} + 3 - \frac{t}{5}$.

26. $\frac{1}{3}(p - 4) - \frac{1}{4}(5 - p) = \frac{1}{6}(3 - 4p)$.

27. $\frac{2x + 3}{3} + \frac{3x + 5}{5} = 6 - \frac{2x - 5}{5}$.

28. $\frac{1}{3}(y + 10) - \frac{1}{6}(y + 1) = \frac{1}{12}(y + 33)$.

29. $\frac{2(2w + 1)}{5} - \frac{1 - w}{4} - \frac{3(3w - 1)}{10} = 0$.

30. $\frac{3t - 1}{2\frac{1}{2}} - \frac{t + 1}{3\frac{1}{3}} = \frac{2t}{1\frac{2}{3}} + 3\frac{1}{2}$.

PART II

CHAPTER IX

SIMULTANEOUS EQUATIONS AND PROBLEMS

Example 1. Two tanks, A and B, contain 120 gallons and 300 gallons of water. If water is allowed to run into A at the rate of 10 gallons per minute, and to run out of B at the rate of 20 gallons a minute, find a formula for the number of gallons n in tank A after t minutes, and a similar formula for tank B.

Water runs into A at 10 gallons per minute;

$\therefore$ in t minutes, $10t$ gallons run into A;

$\therefore$ after t minutes, there are $(120 + 10t)$ gallons in A.

$$\therefore \text{ for tank A, } \mathbf{n = 120 + 10t}.$$

Water runs out of B at 20 gallons per minute;

$\therefore$ in t minutes, $20t$ gallons run out of B.

$\therefore$ after t minutes, there are $(300 - 20t)$ gallons in B.

$$\therefore \text{ for tank B, } \mathbf{n = 300 - 20t}.$$

Since the amount of water in A steadily increases, and that in B steadily decreases, there must come a single moment when A and B contain equal amounts of water. This means that there is one value of t which makes the value of n for tank A the same as the value of n for tank B.

At this moment, *and only at this moment*, the equations

$$n = 120 + 10t; \quad n = 300 - 20t$$

are true for each tank, and we call them **simultaneous equations.**

If the equations $\begin{cases} n = 120 + 10t \\ n = 300 - 20t \end{cases}$ are simultaneous,

then $\qquad 120 + 10t = 300 - 20t; \quad \therefore \ 10t + 20t = 300 - 120:$

$$\therefore \ 30t = 180; \quad \therefore \ t = 6.$$

Also $n = 120 + 10t = 120 + 60 = 180.$

Check : Take the other equation,

$$n = 300 - 20t = 300 - 20 \times 6 = 300 - 120 = 180, \text{ as before.}$$

This shows that in 6 minutes' time there will be equal amounts of water, namely 180 gallons, in the two tanks. At no other time will this be true.

EXERCISE IX. a

1. A and B have £110 and £600 respectively in the bank. If A increases his bank balance by £30 every year and if B decreases his balance by £40 every year, find a formula for A's balance, £P, after n years, and a similar formula for B's balance. Find the time at which their balances are equal, and the amount of the balance at this time.

2. Bodies of various weights are attached to the hooks of the springs in Fig. 147. A stretches $\frac{1}{2}$ inch and B stretches $1\frac{1}{4}$ inches, for each 1 oz. weight attached. Find a formula for the *total* length, l inches, of A when a body of weight W oz. is attached, and a similar formula for B. Find what weight makes the total lengths equal, and what this length is.

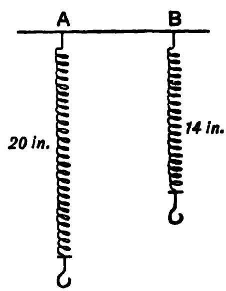


FIG. 147.

3. At one golf Club, there is an entrance fee of £9 and a subscription of £6 a year; at another Club, the subscription is £7. 10s. a year with no entrance fee. Find the formula for the total cost, £C, for n years at each Club. Find also the number of years for which the total cost at the Clubs is the same, and what this cost is.

4. Find the formula for the tax, £T, on an income £P, calculated by each of the following rules, for $P > 200$.

Rule I. No tax on the first £200 of a man's income, and 5s. in the £ on the rest.

Rule II. No tax on the first £160, and 4s. in the £ on the rest.

For what income do the two rules give the same tax, and how much is this tax ?

5. Fig. 148 represents a man A driving from Winchester to Oxford at 20 m.p.h. and a man B driving from Oxford to

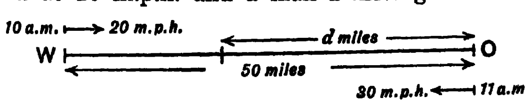


FIG. 148.

Winchester at 30 m.p.h. Find a formula for the distance d miles *from Oxford* at t hours after 10 a.m. (i) for A, (ii) for B.

At what moment are A and B at the same distance from Oxford, (*i.e.* at what time do they meet) and how far is this from Oxford ?

6. Write down three pairs of numbers, x and y, such that $y=2x+1$. Find the only pair of numbers, x and y, for which $y=2x+1$, and such that their sum is 7.

7. Write down three pairs of numbers, p and q, such that $p=\frac{2q}{3}-1$. Find the only pair of numbers, p and q, for which $p=\frac{2q}{3}-1$ and $p=\frac{q}{2}$.

8. Write down three pairs of numbers such that the sum of each pair is 12. Find a pair of numbers whose sum is 12 and whose difference is 6.

Find pairs of values which satisfy the following simultaneous relations :

9. $a=7-2b,$ $a=b-2.$

10. $r=3+s,$ $r=12-s.$

11. $n=2t-3,$ $n=7t-13.$

12. $u=v-3.$ $u=5v-19.$

13. $y=5-3z,$ $y=13+z.$

14. $P=3Q,$ $P=20-2Q.$

15. $\frac{r}{s}=3,$ $r=15-2s.$

16. $b=\frac{1}{3}(a-1),$ $b=\frac{1}{2}(a+1).$

17. $n-3t=2,$ $n+t=14.$

18. $q=r-3,$ $q=7r-21.$

19. $y=\frac{x}{6}+\frac{1}{3},$ $y=\frac{x}{5}.$

20. $2p=5q,$ $p=2(q+1).$

Solution by Substitution

Example 2. Find a pair of numbers, x and y, which satisfy simultaneously the equations,

$$3x+2y=21 \quad\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$2x+5y=3 \quad\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

From (i), $2y=21-3x$, $\therefore\ y=\frac{21-3x}{2}$ .... (iii)

Substitute this value of y in (ii).

Then $2x+\frac{5(21-3x)}{2}=3$; $\therefore\ 4x+105-15x=6$;

$\therefore\ -11x=-99$; $\therefore\ x=9$.

Put $x=9$ in (iii), then $y=\frac{21-27}{2}=-\frac{6}{2}=-3$.

$\therefore$ the pair of numbers is $x=9$, $y=-3$.

Check : If $x=9$ and $y=-3$, $2x+5y=18-15=3$, as in (ii).

Note. The value of y was found by substituting $x=9$ in (iii), which is equivalent to (i). We therefore use (ii) for the check, *not* (i).

EXERCISE IX. b

Find y in terms of x in Nos. 1 9.

1. $2x + y = 7.$
2. $3x - y = 11.$
3. $5x + 2y = 12.$
4. $\frac{x}{2} + 3y + 1 = 0.$
5. $\frac{x}{3} - 4y = 12.$
6. $\frac{x}{2} + \frac{2y}{3} = 10.$
7. $5x + 4y = 0.$
8. $x = \frac{2y - 3}{5}.$
9. $\frac{x}{3} = \frac{y - 1}{4}.$

Find x in terms of y in Nos. 10-15.

10. $3x - 2y = 7.$
11. $\frac{x}{2} + \frac{y}{3} = 0.$
12. $5x - 2y - 7 = 0.$
13. $\frac{x}{4} + \frac{y}{6} + 1 = 0.$
14. $\frac{2x}{3} - \frac{y}{5} = 6.$
15. $\frac{y - 6}{x} = 1\frac{1}{2}.$

Solve the following simultaneous equations, Nos. 16-24.

16. $y = 2x - 1,$ $y = 9 - 2x.$
17. $x = 5 - y,$ $x = 3y - 1.$
18. $r = 5s + 4,$ $r = 2s + 9.$
19. $u = 3v,$ $u = 7 - 2v.$
20. $b = \frac{1}{3}(a - 2),$ $b = \frac{1}{4}(3 - a).$
21. $y = \frac{z}{4} + 3,$ $y = \frac{1}{4}(3 - 5z).$
22. $p = \frac{1}{2}q + 1,$ $2p = 3q - 6.$
23. $d = 5c - 3,$ $3d = 8c + 5.$
24. $\frac{n}{2} = 8 - k,$ $\frac{n}{5} = 3k - 7.$

25. Is it better to find x in terms of y or to find y in terms of x, if you are using the substitution method to solve the following simultaneous equations ? Do not solve them.

(i) $3y - x = 1,$ $7y - 2x = 4.$

(ii) $5y + \frac{x}{4} + 7 = 0,$ $2y - \frac{x}{3} + 8 = 0.$

Solve the following simultaneous equations, Nos. 26-33.

26. $a + 2b = 11,$ $2a - b = 2.$
27. $3p - q = 11,$ $2p - 3q = 5.$
28. $3x - 7y = 35,$ $2x + 5y = 4.$
29. $\frac{y}{2} + \frac{z}{3} = 3,$ $2y + 3z = 2.$
30. $\frac{u}{5} + 2v = 9,$ $\frac{u}{2} - \frac{v}{6} = 7.$
31. $3P = 4Q,$ $\frac{1}{2}P + \frac{2}{5}Q = 4.$
32. $3r - 4s + 2 = 0.$ $3s + r + 1 = 0.$
33. $\frac{1}{2}c - \frac{2}{3}d + 3 = 0.$ $\frac{1}{6}c - \frac{1}{5}d + \frac{1}{4} = 0.$

Solution by Addition or Subtraction

The following method, although applicable to any linear simultaneous equations, is not so general as the substitution method ; it may therefore be omitted, if desired. But in this case Ex. IX. c should be used for further practice in the substitution method. The general instructions on p. 150 apply to both methods.

Example 3. Solve the simultaneous equations

$$3x - 2y = 11 \quad \text{(i)}$$
$$5x + 2y = 29 \quad \text{(ii)}$$

The result of adding the left side of (i) to the left side of (ii) equals the result of adding the right sides ; but if we do this, the term in y disappears, leaving a simple equation in x.

Thus, $3x + 5x = 11 + 29$; $\therefore\ 8x = 40$;

$$\therefore\ x = 5.$$

Put $x = 5$ in (i), then $15 - 2y = 11$; $\therefore\ -2y = -4$;

$$\therefore\ y = 2.$$

$$\therefore \text{ the solution is } x = 5,\ y = 2.$$

Check : If $x = 5$ and $y = 2$, $5x + 2y = 25 + 4 = 29$, as in (ii).

Example 4. Solve the simultaneous equations

$$2x - 5y = 27 \quad \text{(i)}$$
$$2x + 3y = 3 \quad \text{(ii)}$$

If we subtract, the term in x disappears.

Subtracting, $-5y - 3y = 27 - 3$; $\therefore\ -8y = 24$;

$$\therefore\ y = -3.$$

Put $y = -3$ in (ii), then $2x - 9 = 3$; $\therefore\ 2x = 12$;

$$\therefore\ x = 6.$$

$$\therefore \text{ the solution is } x = 6,\ y = -3.$$

Check : If $x = 6$ and $y = -3$, $2x - 5y = 12 + 15 = 27$, as in (i).

The process of getting rid of one of the unknowns is called **elimination.** In Example 3, we eliminated y ; and in Example 4, we eliminated x. It does not matter which unknown is eliminated, always do whichever is easier.

Example 5. Solve the simultaneous equations

$$7x - 6y = 20 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$3x + 4y = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

The L.C.M. of 6 and 4 is 12 ; $\therefore$ if we multiply each side of (i) by 2 and each side of (ii) by 3, we shall obtain equations in which the coefficients of y are *numerically* equal.

Multiply each side of (i) by 2, $\quad\therefore\ 14x - 12y = 40$ ....(iii)
Multiply each side of (ii) by 3, $\quad\therefore\ 9x + 12y = 6$ ....(iv)
From (iii) and (iv) by adding, $\quad 23x = 46$; $\quad\therefore\ x = 2$.
Put $x = 2$ in (ii), $\quad\therefore\ 6 + 4y = 2$; $\quad\therefore\ 4y = -4$; $\quad\therefore\ y = -1$.

$\therefore$ the solution is $x = 2$, $y = -1$.

Check : If $x = 2$ and $y = -1$, $7x - 6y = 14 + 6 = 20$, as in (i).

General Instructions

(i) First decide which unknown it is easier to eliminate.

(ii) When one unknown has been found, obtain the other by substituting in the EASIEST equation you have, containing it.

(iii) When checking, use that one of the ORIGINAL equations, which is not equivalent to the equation used for substitution.

(iv) If the answers involve awkward fractions or decimals, it is often better to look over your working again or solve by an independent method, instead of checking by substitution.

(v) Number the chief equations, and use the numbering to explain your method.

EXERCISE IX. c

Solve the following pairs of simultaneous equations and check your answers.

In each case, write down *at the start* which unknown can be eliminated the more easily, or whether it makes no difference.

1. $a + b = 11$, $a - b = 5$. **2.** $3p + q = 11$, $p + q = 7$. **3.** $2u - v = 3$, $u + v = 9$.

4. $r + 3s = 8$, $r - 2s = 3$. **5.** $x - 5y = 1$, $x + 4y = 28$. **6.** $2y - 3z = 13$, $2y - 4z = 10$.

7. $c - 3d = 0$, $2c - d = 20$. **8.** $p = 2 + q$, $q = 8 - p$. **9.** $l + 2m = 8$, $2l + m = 7$.

10. $3s - t = 1$, $5s + 2t = 20$. **11.** $a - 2b = 3$, $3a + b = 72$. **12.** $y - 2z = 5$, $2z + y = 1$.

13. $x-2y=27$, $7x+y=9$.

14. $d=3c-2$, $c=1-2d$.

15. $2y=6z+1$, $2z=3-2y$.

16. $2l+m=10$, $3l-2m=1$.

17. $6p-5q=24$, $9p-4q=22$.

18. $3x+4y+11=0$, $5x+6y+7=0$.

19. $3a-5b=4$, $2(3-a)=3+a$.

20. $2(r-2s)=3s$, $3s-2r=1$.

21. $2(x+2)=6(y+1)$, $2x-5y=4$.

22. $P+13Q=19$, $P-11Q=67$.

23. $6a-5b=21$, $5a+4b=17\frac{1}{2}$.

24. $10c+10d=7$, $3d-c=\frac{1}{10}$.

25. $4x-2y=4y-5x=x+y-3$.

26. $3p-1=10p+q=1\frac{1}{2}p$.

27. $2y-3z=6y+5z=70$.

28. $3l-8=l-2m-1=40$.

29. $5a-3b=7a-6b=9$.

30. $x-y=y-x+2=3y$.

31. $2c-d-3=3d-c+4=5c-8d-4$.

32. $4x-6y-3=7x+2y-4=3y-2x+24$.

33. $2(a-1)-3(b-3)=12$; $5(a-1)+3(b-3)=9$.

34. Find a pair of numbers satisfying $7x-2y=38$, and such that one is three times the other. Is there more than one answer ?

35. If $y=\dfrac{22x-45}{13}$ and if x and y are equal numbers, find their values.

36. If $2a+3b=9$ and $3a+2b=16$, find the value of $3a-2b$.

37. If $3p+4q=-119$ and $5p-11q=102$, prove that $p=q$.

38. If $y=mx+b$ is satisfied by $x=2$, $y=3$ and also by $x=-4$, $y=1$, find the values of m and b.

39. If $x=3$, $y=-2$, satisfy simultaneously the equations $x+ay=5$, $bx+y=7$, find the values of a, b.

40. If $x=my+a$ is satisfied when $x=2$, $y=3$ and also when $x=-5$, $y=4$, find the values of m, a. Find also the value of y when $x=9$.

[*Note. For additional drill-examples, see Exercise E.P.* 14, *p.* 249.]

Problems

Example 6. A certain number is formed of two digits ; its value equals four times the sum of its digits. If 27 is added to it, the sum is the number obtained by interchanging the digits ? What is the number ?

Let x be the tens-digit and y the unit-digit. Then the value of the number is $10x+y$.

$$\therefore\ 10x+y=4(x+y)\,;$$

$$\therefore\ 10x+y=4x+4y\,;\quad \therefore\ 6x-3y=0\,;$$

$$\therefore\ 2x-y=0 \quad \text{.............................(i)}$$

The value of the number formed by interchanging the digits is $10y+x$.

$$\therefore\ 10x+y+27=10y+x\ ;\quad \therefore\ 9x-9y=-27\ ;$$

$$\therefore\ x-y=-3 \text{.............................(ii)}$$

From (i) and (ii) by subtracting, $x=3$.

$$\therefore\ \text{from (i), } 6-y=0\ ;\quad \therefore\ y=6.$$

$$\therefore\ \text{the original number is 36.}$$

Check by using the data of the problem :

$4\times$(sum of digits) $=4\times(3+6)=4\times 9=36=$ the number.

$36+27=63=$ number obtained by interchanging the digits.

EXERCISE IX. d

Solve the following problems by means of *simultaneous* equations.

1. Find a pair of numbers whose sum is 39 and whose difference is 11.

2. Fig. 149 shows the lengths of the sides of a triangle in inches. If the triangle is equilateral, find its perimeter.

3. 5 lb. of apples and 2 lb. of pears cost 4s. ; 1 lb. of apples and 2 lb. of pears cost 2s. Find the cost of 1 lb. of pears.

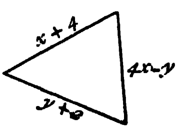


FIG. 149.

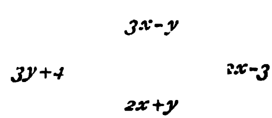


FIG. 150.

4. A gramophone and 20 records cost £8. 10s. ; the same gramophone and 50 similar records cost £12. 5s. What does the gramophone cost by itself ?

5. A farmer can buy 3 cows and 5 sheep for £90, or 4 cows and 10 sheep for £140. What is the price of a cow and of a sheep ?

6. Find the length and breadth of the rectangle in Fig. 150, the units of the data being inches.

7. 2 knives and 4 forks cost £1. 4s. ; 5 knives and 6 forks cost £2. Find the price of a knife and of a fork.

8. The external perimeter of the network in Fig. 151 is 20 in., and the total length of all the lines forming it is 3 ft. ; find the external length and breadth.

FIG. 151.

9. A builder requires 3 lorry loads and 8 cart loads to fetch 15 tons of gravel. He would require 2 lorry loads and 20 cartloads for 21 tons. How much is a lorry load and a cartload ?

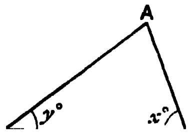


FIG. 152.

10. In Fig. 152, $\angle BAC = 2\angle ABC$ and $\angle ACB - \angle ABC = 36°$; find the angles of the triangle.

11. In Fig. 152, $\angle BAC$ can be expressed either as $(2x-y)$ degrees or as $(2y+x)$ degrees. Find the angles of the triangle.

12. I am thinking of a pair of numbers. If I add 11 to the first, I obtain twice the second ; and if I add 20 to the second, I obtain twice the first. What are the numbers ?

13. A bench 18 ft. long holds either 3 men and 8 boys or 6 men and 4 boys. What length of bench is required for 10 men and 10 boys ?

14. A piece of wire, $6\frac{1}{2}$ ft. long, can be bent into either of the

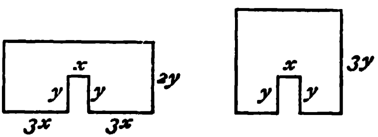


FIG. 153.

shapes in Fig. 153, the units of the data being inches, and all the corners right-angled. Find x and y.

15. If A gives B 3 shillings, B will have twice as much as A ; if B gives A 5 shillings, A will have twice as much as B. How much has each ?

16. The area of Fig. 154 is $\frac{1}{2}$ sq. ft., and the perimeter is 4 ft. ;

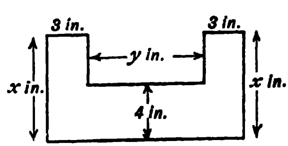


FIG. 154.

find x and y.

17. Find a fraction which reduces to $\frac{2}{3}$ if the numerator and denominator are each increased by 1, and reduces to $\frac{3}{5}$ if the numerator and denominator are each diminished by 2.

18. A light rod is supported by loops attached to two spring balances, and a heavy load W lb. is attached to the rod, see

Fig. 155. If the readings on the spring balances are P lb., Q lb., it is known that $P + Q = W$ and $a \cdot P = b \cdot Q$.

(i) Find P and Q if $W = 20$, $a = 10$, $b = 6$.

(ii) Find P and W if $Q = 9$, $a = 8$, $b = 12$.

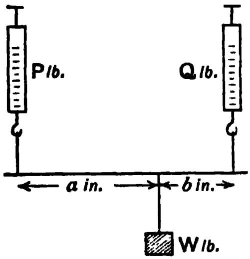


FIG. 155.

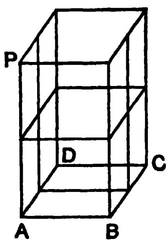


FIG. 156.

19. A skeleton cuboid, see Fig. 156, has a square base ABCD; the wire composing the cuboid is 11 ft. long, and of this, 7 ft. is used for the outside edges, *i.e.* all the rims. Find the dimensions of the cuboid.

20. A number of two digits is equal to 7 times the sum of the digits and exceeds by 36 the number formed by interchanging the digits. What is the original number?

21. If the larger of two numbers is divided by the smaller, the quotient and remainder are each 2; if 5 times the smaller is divided by the larger, the quotient and remainder are again each 2. What are the numbers?

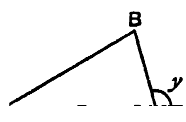


FIG. 157.

22. In Fig. 157, $AB = AC$ and $\angle ABC = \frac{1}{3}(x + 2y)$ degrees; find x and y.

23. A heap of florins and half-crowns is worth £2. If there were 3 times as many florins and half the number of half-crowns, it would be worth 5s. more. How many coins of each kind are there?

24. If in Fig. 158, $AC = AB = BD$, find one equation connecting x and y. If further $BC = CD$, find x and y.

25. A man travels 12 miles in $2\frac{1}{4}$ hours; part of the time he is driving at 9 miles an hour and the rest of it he is walking at $3\frac{1}{2}$ miles an hour. How far did he drive and how far did he walk?

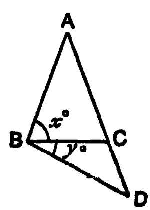


FIG. 158.

[*Note. For additional easy problems, see Exercise E.P.* 15, *p.* 250. *For additional harder problems, see Appendix, Ex. S.* 8, *p.* 294.]

Example 7. Solve the simultaneous equations

$$\frac{2x-1}{3}-\frac{3(y+1)}{5}=1\tfrac{1}{2} \qquad \text{.........................(i)}$$

$$\frac{2x-y}{2}-\frac{y-1}{3}=4\tfrac{1}{12} \qquad \text{.........................(ii)}$$

Multiply each side of (i) by 15,

$\therefore\ 5(2x-1)-9(y+1)=1\tfrac{1}{2}\times 15$;

$\therefore\ 10x-5-9y-9=22\tfrac{1}{2}$;

$\therefore\ 10x-9y=36\tfrac{1}{2}$ ....(iii)

Multiply each side of (ii) by 6,

$\therefore\ 3(2x-y)-2(y-1)=4\tfrac{1}{12}\times 6$;

$\therefore\ 6x-3y-2y+2=24\tfrac{1}{2}$;

$\therefore\ 6x-5y=22\tfrac{1}{2}$ ....(iv)

Multiply each side of (iii) by 3, and each side of (iv) by 5.

From (iii), $30x-27y=36\tfrac{1}{2}\times 3=109\tfrac{1}{2}$ ....(v)

From (iv), $30x-25y=22\tfrac{1}{2}\times 5=112\tfrac{1}{2}$ ....(vi)

From (v) and (vi), by subtraction, $-2y=-3$;

$\therefore\ y=1\tfrac{1}{2}$.

Substitute in (iv) for y, $\therefore\ 6x-5\times 1\tfrac{1}{2}=22\tfrac{1}{2}$;

$\therefore\ 6x=22\tfrac{1}{2}+7\tfrac{1}{2}=30$; $\therefore\ x=5$.

$\therefore$ the solution is $\mathbf{x}=5$, $\mathbf{y}=1\tfrac{1}{2}$.

The reader should now consider which of the equations, (i) or (ii), should be used for checking.

When x or y have fractional (or decimal) coefficients, it is usually best to start by clearing these terms of fractions ; *it is not necessary to clear of fractions the terms which do not contain x or y.*

EXERCISE IX. e

Solve the following pairs of simultaneous equations :

1. $x+\frac{y}{2}=13$,
   $\frac{x}{3}-y=2$.

2. $\frac{a}{2}-2b=5$,
   $\frac{a}{3}+b=1$.

3. $\frac{3p}{4}-\frac{q}{3}=9$.
   $2q-p=2$.

4. $\frac{2y}{5}-\frac{z}{3}=2\tfrac{2}{3}$.
   $y=2(z+1)$.

5. $\frac{r}{3}+\frac{t}{5}=8$,
   $\frac{r}{9}-\frac{t}{10}=1$.

6. $y-\frac{2x}{7}=\tfrac{1}{3}$,
   $x-y=3\tfrac{1}{6}$.

7. $\frac{1}{2}u+2v=5,$
$2u-\frac{v}{7}=-8\frac{1}{2}.$

8. $2c+5d=1\frac{1}{2},$
$9c-7d=\frac{17}{20}.$

9. $Q+2=\frac{1}{3}P,$
$\frac{P}{5}-1=\frac{2}{3}Q.$

10. $\frac{x+1}{2}=8-\frac{y-1}{3},$
$\frac{y+1}{2}=9-\frac{x-1}{3}.$

11. $\frac{y+1}{3}+\frac{z-1}{2}=5,$
$\frac{2y+5}{3}-\frac{z+1}{4}=3.$

12. $l-5=\frac{1}{7}(n-2),$
$4n-3=\frac{1}{3}(l+10).$

13. $p=\frac{1}{2}-3(q-\frac{3}{4}),$
$p=\frac{1}{3}q+\frac{1}{4}.$

14. $\frac{r}{10}+\frac{s}{8}=r-s,$
$\frac{2r-s}{3}+2s=\frac{1}{2}.$

15. $\frac{a-1}{2}+\frac{b+1}{5}=4\frac{1}{5},$
$\frac{a+b}{3}=b-1.$

16. $\frac{1}{3}(R+2)-\frac{1}{4}(r-1)=1,\ r=1\frac{1}{4}R.$

17. $\frac{y+1}{3}-\frac{3z-1}{2}=1,\quad \frac{3-8z}{5}-\frac{7-3y}{4}=1.$

18. $1{\cdot}2x-0{\cdot}8y=0{\cdot}4,$
$y=0{\cdot}3-0{\cdot}1x.$

19. $0{\cdot}5t-0{\cdot}7v=2,$
$1{\cdot}8t-2{\cdot}2v=8\ 8.$

20. $1{\cdot}25a-0{\cdot}5b=0{\cdot}25,$
$0{\cdot}2a+0{\cdot}1b=1{\cdot}3.$

21. $h+5{\cdot}2n=12{\cdot}1.$
$h+3{\cdot}8n=10.$

22. $\frac{1}{8}(4y+5z-9)=5y-5z,\ \frac{1}{3}(2y-z-1)=2\frac{1}{2}-2z.$

23. $3a+\frac{1}{2}b=8a+7b-9=2.$

24. $\frac{x}{2}-\frac{y}{3}=\frac{2x}{5}-\frac{y}{4}=3x-2y-5.$

25. $\frac{1}{6}(5p+2q)=p+q=2p+3q+1.$

26. $\frac{1}{5}(c-3)+\frac{1}{3}(d-7)=c-d=\frac{1}{3}(5c+d).$

27. $r+s-8=\frac{1}{2}(r-s)+\frac{2}{3}\left(r-\frac{s}{2}+2\right)=0.$

28. $\frac{b+1}{3}-\frac{c+1}{2}=2b+c+1,\ \frac{1}{3}(b-2)=\frac{1}{2}(c+3).$

29. $\frac{x-2}{5}-\frac{y-3}{4}=2\frac{3}{4},\ \frac{3-x}{4}=\frac{y-1}{5}.$

30. $2l-\frac{1}{5}(m-3)=9,\ 3m+\frac{1}{3}(l-2)=25.$

[*Note. For additional equations, see Appendix, Ex. S. 9, p. 297.*]

CHAPTER X

GRAPHS OF FUNCTIONS

Example 1. Draw a graph representing the squares of numbers from 0 to 5.

Make a table connecting the value of any number x and its square, x^2, from 0 to 5.

0	1	2	3	4	
0	1	4	9	16	25

For the scale of numbers (values of x), let 1 inch represent 2 units.

For the scale of squares (values of x^2), let 1 inch represent 10 units.

Plot the points which represent the values in the above table.

The last two points are rather far apart; we therefore take additional values to make the drawing of the graph easier;

x	3·5	4·5
x^2	12·25	20·25

add these to the table, and plot the corresponding points.

Now draw a smooth curve through the plotted points. This gives Fig. 159, see p. 158.

The graph may now be used to read off (approximately) squares of numbers between 0 and 5. Thus, to obtain the square of 2·6, take, on the axis across the page, the point corresponding to $x = 2 \cdot 6$, and note the reading for the upright from this point to the curve, approximately 6·8, $\therefore\ 2 \cdot 6^2 \simeq 6 \cdot 8$.

Since this graph represents the relation between a number x and its square x^2, it is called the **graph of the function $\mathbf{x^2}$.**

Any expression, containing x, whose value can be found when the value of x is given, is called a **function of** x. Thus $7x$, $\frac{3}{4}x - 5$, $\frac{2x-1}{x+3}$, $x^3 - 5x$, etc., are all functions of x. The letter y is generally used to represent any function of x. Thus, we call

Figure 159 the graph of $y = x^2$, for values of x from 0 to 5; *the axis across the page,* along which values of x are measured, *is called the x-axis*; *the axis up the page,* along which values of y (where $y = x^2$) are measured, *is called the y-axis.* The point of intersection of the axes is called the origin and is usually denoted by O.

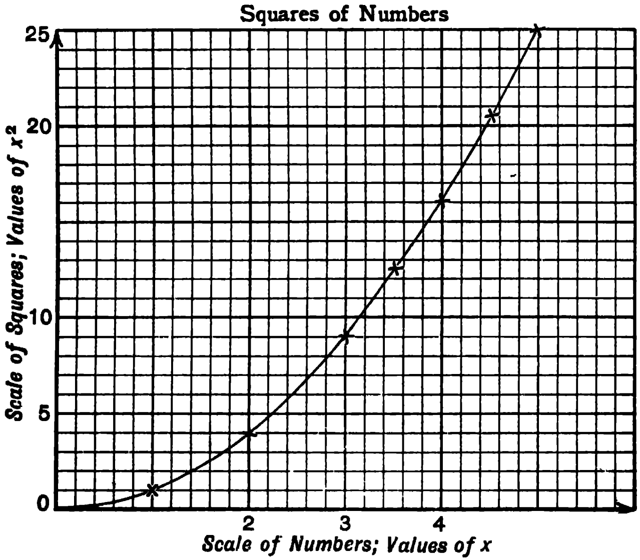


FIG. 159.

Oral Work on Fig. 159.

(i) Read off from the graph the squares of 3·8, 2·4, 4·4.

(ii) Use the graph to find approximately a value of x such that $x^2 = 10$.

(iii) Read off the square roots of 23, 5, 14.

Example 2. Draw the graph of the function $y = x^2$ for values of x from -4 to $+4$.

Make a table of values.

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$	16	9	4	1	0	1	4	9	16

Regard the x-axis as a number-scale, points to the right of O representing positive numbers; then points to the left of O

represent negative numbers. Using the same scales as before,

Graph of $y = x^2$

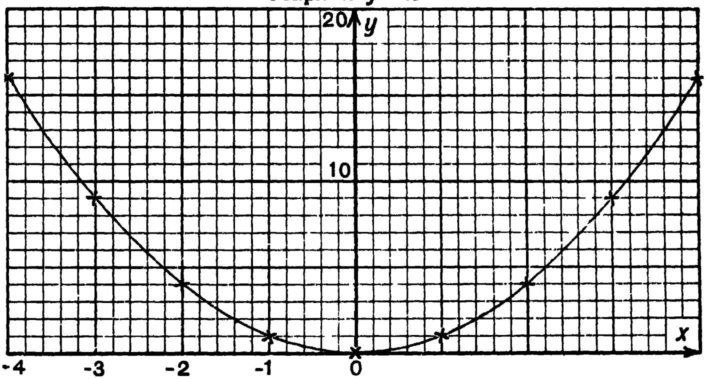


FIG. 160.

plot the points which represent the table of values, and draw the graph, see Fig. 160, (slightly reduced).

Example 3. A marble is projected up a sloping groove and moves so that it is passing now (zero hour, $t = 0$) a marked point A, and t seconds later is s feet up the groove from A, where $s = 20t - 5t^2$. Draw a graph showing the position of the marble from 2 seconds ago ($t = -2$) to 6 seconds ahead ($t = +6$).

FIG. 161.

In making a table of values, proceed as follows: *Write down the whole of a row,* before beginning the next row.

First write down the row of selected values of t, then the row of values of $20t$. *Before* writing down the row of values of $-5t^2$, put down the row of values of t^2 *above* the table to help you to find $-5t^2$.

t^2	4	1	0	1	4	9	16	25	36
t	-2	-1	0	1	2	3	4	5	6
$20t$ $-5t^2$	-40 -20	-20 -5	0 0	20 -5	40 -20	60 -45	80 -80	100 -125	120 -180
$20t - 5t^2$	-60	-25	0	15	20	15	0	-25	-60

Since $20t - 5t^2 = 5t(4 - t)$, the table of values can be made by the following *alternative method* :

t	-2	-1	0	1	2	3	4	5	6
$5t$ $4 - t$	-10 6	-5 5	0 4	5 3	10 2	15 1	20 0	25 -1	30 -2
$5t(4 - t)$	-60	-25	0	15	20	15	0	-25	-60

As before, points to the right of the origin on the t-axis represent positive numbers, those to the left of it, negative numbers. Similarly on the s-axis, points above the origin represent positive numbers, and points below the origin, negative numbers.

We choose scales as follows : for the t-axis, 1 inch represents 2 seconds ; for the s-axis, 1 inch represents 40 feet.

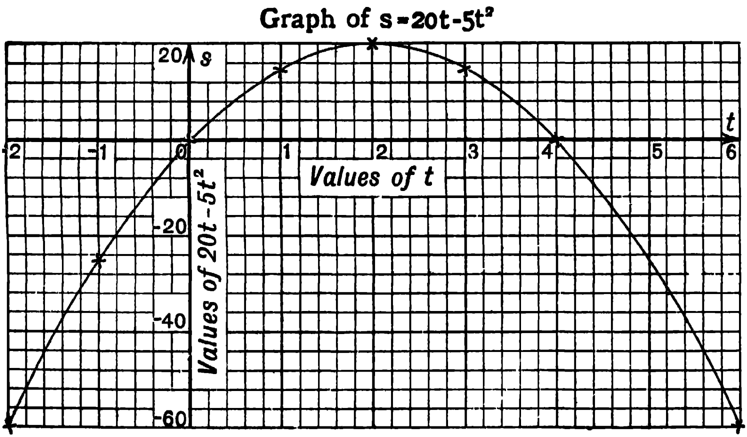


FIG. 162.

Plotting the values of s or $20t - 5t^2$ against the values of t, we obtain Fig. 162.

EXERCISE X. a

Use Fig. 162 to answer the following questions :

1. What is the value of s when $t = 3{\cdot}8$, $4{\cdot}4$, $-0{\cdot}6$?

2. How far above or below A is the marble when $t = 0{\cdot}6$, $4{\cdot}8$, $-1{\cdot}4$, $3{\cdot}5$, $2{\cdot}6$, $1{\cdot}4$, $-1{\cdot}8$? What do these values of t mean ?

3. What is the highest point the marble reaches and at what time ?

4. What can you say about the value of t if $s = 8, \ -12, \ -36$?

5. When is the marble at P, see Fig. 161, if AP is (i) 4 feet, (ii) 14 feet, (iii) 20 feet ?

6. When is the marble at Q, see Fig. 161, if AQ is (i) 16 feet, (ii) 32 feet, (iii) 52 feet, (iv) 60 feet ?

7. At what time does the marble pass A, coming down ?

8. (i) For what values of t is $20t - 5t^2$ equal to 14 ?
(ii) Solve the equation $20t - 5t^2 = 10$.

9. Solve the equations : (i) $20t - 5t^2 = 4$; (ii) $20t - 5t^2 = -14$; (iii) $20t - 5t^2 = -40$; (iv) $5t^2 - 20t + 8 = 0$.

10. Is there a value of t for which $20t - 5t^2 = 25$? What is the connection between your answer and the movement of the marble ?

11. If, in Fig. 161, AQ is 60 feet, how long is it between the times when the marble passes Q going up and coming down ?

12. For what values of t is s positive ? For what values of t is s greater than -20 ?

13. How far does the marble move between the times, $t = 0{\cdot}6$ and $t = 1$? What is its average velocity in this interval ?

14. Interpret the fact that the graph is (i) steeper at $t = -2$ than at $t = 1$; (ii) flat at $t = 2$.

EXERCISE X. b

1. A stone projected vertically upwards with velocity 40 ft. per sec. is s feet above its starting point after t seconds where $s = 8t(5 - 2t)$.

Represent this function by a graph for values of t from 0 to 3.

For scales, along the t-axis let 1 inch represent $\frac{1}{2}$ unit ; along the s-axis let 1 inch represent 10 units. Start by making a table of values for $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}$, etc. After plotting these values, you will see that the graph can be drawn more accurately if you add to the table $t = 1\frac{1}{4}, 2\frac{3}{4}$, and two other values.

Use your graph to answer the following questions :

(i) What is s when $t = 0{\cdot}8, 1{\cdot}9, 2{\cdot}6$?
(ii) For what values of t is $s = 11, 20, 22$?
(iii) What is the greatest height the stone reaches and after what time ?
(iv) How long is the stone in the air ?
(v) At what times is the stone 15 feet above the ground ?

(vi) For how long is the stone more than 12 feet above the ground ?

(vii) What meaning can you give to the part of the graph for which $t > 2 \cdot 5$?

(viii) For what values of t is $8t(5-2t)$ equal to 5 ? Solve the equation $40t - 16t^2 = 7$.

(ix) Solve the equations : (a) $8t(5-2t) = 21$; (b) $40t - 16t^2 = 15$.

(x) Find one value of t for which $8t(5-2t) = -15$.

(xi) Is there a value of t for which $8t(5-2t) = 27$?

(xii) How far does the stone go between the times $t = \frac{1}{4}$ and $t = \frac{1}{2}$? What is its average velocity in this interval ? The curve is steeper at $t = \frac{1}{2}$ than at $t = 1$, what does this mean ?

2. Draw the graph of the function $y = (x+1)(x-2)$ for values of x from -3 to $+4$. Use it to answer the following :

(i) What is y when $x = 2 \cdot 4,\ 1 \cdot 6,\ -2 \cdot 6$?

(ii) For what values of x is $y = 6,\ -1 \cdot 5$?

(iii) What is the least value of $(x+1)(x-2)$? For what value of x is it least ?

(iv) For what values of x is $(x+1)(x-2)$ equal to 8 ? Solve the equation $(x+1)(x-2) = 5$.

(v) Solve the equations, (a) $(x+1)(x-2) = 3$;
(b) $(x+1)(x-2) = -1$.

(vi) Is there a value of x for which $(x+1)(x-2) = -3$?

(vii) For what values of x is $(x+1)(x-2)$ less than 7 ?

3. The manager of a shop finds that the profits per week depend on how he divides his time between the shop and the office. If he spends n hours per day in the shop, the weekly profits £p are given by the relation, $p = 11 + 24n - 3n^2$.

Represent this function by a graph for values of n from 0 to 8, and use it to answer the following :

(i) Is it more profitable for the manager to be in the shop $2\frac{1}{2}$ hours or $3\frac{1}{2}$ hours ? $2\frac{1}{2}$ hours or 6 hours ?

(ii) What is the most profitable length of time for him to be in the shop ?

(iii) What is the profit if he stays in the shop $1\frac{1}{2}$ hours, $5\frac{1}{2}$ hours ?

(iv) How long is he in the shop if the profit is £20, £40, £50 ?

(v) Solve the equation, $11 + 24n - 3n^2 = 36$.

(vi) What is the greatest value of $24n - 3n^2$?

4. Draw the graph of the function $y = x^3$ for values of x from 0 to 5.

Read off from the graph (i) the cubes of $2 \cdot 3,\ 3 \cdot 8,\ 4 \cdot 7$; (ii) the cube roots of 20, 36, 74.

For what range of values of x is x^3 less than 50 ? Find a few values of x^3 when x is negative, then draw freehand on plain paper a rough *sketch* of the graph of $y = x^3$ between $x = -5$ and $x = +5$.

5. Draw the graph of the function $y = \frac{1}{x}$ for values of x from 0·2 to 10.

Find from the graph the reciprocals of 1·3, 4·8, 6·7.

$$\left[\text{The reciprocal of a number N is } \frac{1}{\text{N}}\right].$$

Calculate the value of y when $x = 0{\cdot}1$, when $x = 0{\cdot}01$, when $x = 0{\cdot}001$. How does the graph behave when x approaches 0? Calculate the value of y when $x = 50$ and when $x = 500$. How does the graph behave when x is large and positive?

Find a few values of $\frac{1}{x}$ when x is negative, then draw freehand on plain paper a rough *sketch* of the graph of $y = \frac{1}{x}$ from $x = -10$ to $x = -0{\cdot}2$ and from $x = 0{\cdot}2$ to $x = 10$.

6. A leaden sheet, 30 inches wide is bent to form an open gutter of rectangular section, see Fig. 163.

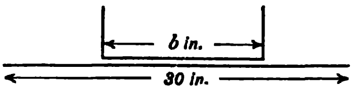


FIG. 163.

If the area of the cross-section is A sq. in., when the width of the gutter is b in., express A as a function of b.

Represent the relation between A and b by a graph, for values of b from 3 to 27; hence answer the following:

(i) What is the maximum area of the cross-section?
(ii) For what values of b is the area of the cross-section 90 sq. in., 100 sq. in.?
(iii) For what values of b is the area of the cross-section greater than 80 sq. in.?
(iv) Solve the equations, (*a*) $\frac{1}{2}x(30 - x) = 60$; (*b*) $x(30 - x) = 190$.

7. The length of a degree of longitude in latitude $x°$ is y miles, where $y = 69 - \frac{x^2}{100}$, approximately. Draw the graph of this function for values of x from 0 to 60. What is the length of a degree of longitude through London (latitude $51\frac{1}{2}°$) and through Rome (latitude 42°)? In what latitude is a degree of longitude 63 miles long?

8. Draw the graph of the function $y = (x + 3)(2 - x)$ for values of x from -5 to $+4$.

What is the greatest value of $(x + 3)(2 - x)$?

Solve the equations:

(i) $(x + 3)(2 - x) = 5$; (ii) $(x + 3)(2 - x) = 2$;
(iii) $(x + 3)(2 - x) = -10$; (iv) $(x + 3)(2 - x) = 0$.

9. Draw the graph of the function $y = 4x^2 + 6x - 7$ for values of x from -3 to $+2$.

(i) Has it a greatest or a least value? How much is it?
(ii) For what values of x is $4x^2 + 6x - 7$ less than 5?
(iii) For what values of x is $4x^2 + 6x - 7$ equal to -5?
(iv) Solve the equation, $4x^2 + 6x - 7 = 0$.

10. From a sheet of cardboard, see Fig. 164, equal squares, side x inches, are cut away at each corner. A box is then formed

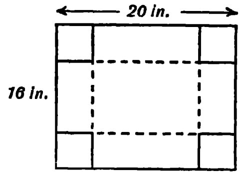


FIG. 164.

by folding along the dotted lines. Express the volume, V cu. in., of the box as a function of x, and represent the relation between V and x graphically for values of x from 0 to 8.

For what value of x is the volume of the box greatest?

Functions of the First Degree

The graph of the function, $y = 3x - 2$. Consider the following table of values:

x	-2	-1	0	1	2	3	4
$3x - 2$	-8	-5	-2	1	4	7	10

We have selected values of x which increase by 1, and we find that the corresponding values of $3x - 2$ increase by equal amounts, namely 3. This means that *the graph has the same slope throughout and must therefore be a straight line.* It corresponds to the fact that if you slide down a straight staircase on a tea-tray, and if all the steps have the same width and the same height, you will move in a straight line.

The values in the table are plotted in Fig. 165; they lie on the straight line AB.

The same argument applies to every first-degree function of x; since its graph is a straight line, it is called a **linear function** of x.

Thus, any function of the form $bx+c$, where b, c are constants, is called a *linear function* of x, and its graph can be drawn by

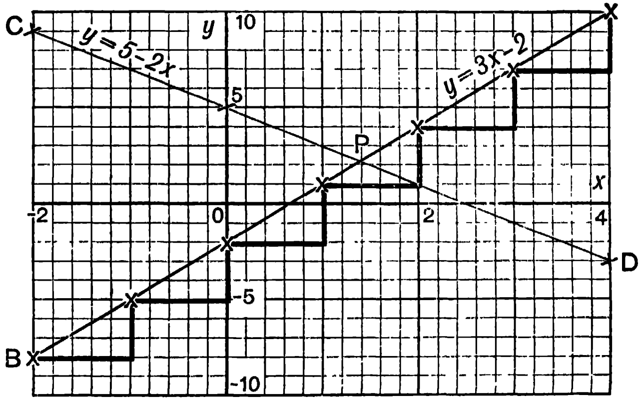


FIG. 165.

plotting two points only, and drawing the straight line which joins them; it is, however, advisable to plot a third point as a check.

Example 4. Find from the graph the value of x for which the functions $5-2x$ and $3x-2$ are equal.

x	-2	0	2	4
$5-2x$	9	5	1	-3

We have selected values of x which increase by 2, and we find that the corresponding values of $5-2x$ decrease by equal amounts, namely 4. This means that *the graph has the same slope throughout (here a downward slope) and must therefore be a straight line.*

The graph is the straight line CD in Fig. 165.

From the figure, we see that the two functions have the same value when $x=1{\cdot}4$, and this value is $2{\cdot}2$.

Check:

When $x=1{\cdot}4$, $5-2x=5-2{\cdot}8=2{\cdot}2$; $3x-2=4{\cdot}2-2=2{\cdot}2$.

Or, solve $5-2x=3x-2$.

Coordinates. The position of a point in a plane is described most easily as follows:

If Ox, Oy are two perpendicular lines (see Fig. 165) called the x-axis and the y-axis, the position of the point A is fixed by saying that if we start from O and move 4 units x-wards and 10 units y-wards, we arrive at A; we call 4 the *x-coordinate* of A and 10 the *y-coordinate* of A, and we speak of A as the point (4, 10). **The x-coordinate is always named first.** Similarly, to arrive at C, we start from O and move (-2) units x-wards and then $(+9)$ units y-wards; $\therefore$ C is the point $(-2, 9)$. Similarly B is the point $(-2, -8)$ and D is the point $(4, -3)$.

Graphs and Equations. The graph of the function, $y=3x-2$, was obtained, see Fig. 165, by plotting the points $(-2, -8)$, $(-1, -5)$, $(0, -2)$, $(1, 1)$, $(2, 4)$, $(3, 7)$, $(4, 10)$ and drawing the straight line AB which passed through them. We call $y=3x-2$ the *equation of this straight line*, because each pair of values of x and y which satisfy the equation, $y=3x-2$, are the coordinates of a point on the line AB.

Similarly, $y=5-2x$ is the equation of the line CD in Fig. 165. The coordinates of the point of intersection P of the lines AB and CD therefore satisfy both equations, $y=3x-2$ and $y=5-2x$. Hence if we read off from the graph the coordinates of P, $x=1{\cdot}4$, $y=2{\cdot}2$, we obtain the solution of the *simultaneous* equations, $y=3x-2$; $y=5-2x$. For another example, see p. 238, fig. 195.

The equation of the curve in Fig. 160 is $y=x^2$ and the equation of the curve in Fig. 162 is $y=20x-5x^2$; these curves are called parabolas. Any function of the second degree in x is of the form, ax^2+bx+c, and is called a **quadratic function** of x; and the graph of $y=ax^2+bx+c$ is shaped as in Fig. 160 or Fig. 162.

EXERCISE X. c

1. Draw the graphs of the functions, $y=2x+3$ and $y=9-3x$, from $x=-2$ to $x=+3$.
   For what value of x are the functions $2x+3$ and $9-3x$ equal?
   Solve the simultaneous equations $y=2x+3$; $y=9-3x$.

2. (i) The values of $5x-7$ are calculated for values of x increasing by 3, *e.g.* $x=1, 4, 7, 10$, etc.; how do the corresponding values of $5x-7$ change? How is this shown in the graph of $5x-7$?
   (ii) Answer the same question for the function $11-5x$.

3. Draw on the same figure the graphs of

(i) $y=2x$; (ii) $y=3x$; (iii) $y=4x$; (iv) $y=-2x$.

What do you notice about all of them ?

What can you say about the graph of $y=bx$, where b is a constant ?

What also, if b is positive ? if b is negative ?

4. Draw on the same figure the graphs of

(i) $y=2x+5$; (ii) $y=3x+5$; (iii) $y=4x+5$; (iv) $y=-2x+5$.

What do you notice about all of them ?

What can you say about the graph of $y=bx+5$, where b is a constant ?

What also, if b is positive ? if b is negative ?

Solve graphically the following simultaneous equations, and compare by solving algebraically.

5. $y=2x-1$, $y=9-2x$.

6. $y=5x+4$, $y=2x+9$.

7. $y=\frac{1}{3}(x-2)$, $y=\frac{1}{4}(3-x)$.

8. $y=3x$, $y=7-2x$.

9. $y=\frac{1}{4}(3-5x)$, $y=\frac{x}{4}+3$.

10. $y=\frac{1}{3}(2x+6)$. $y=-\frac{1}{2}(x+5)$.

11. (i) If $3x+2y=6$, prove that $y=3-\frac{3x}{2}$;

(ii) If $4y-4x=7$, express y in terms of x;

(iii) Solve graphically the simultaneous equations,

$$3x+2y=6\ ; \qquad 4y-4x=7.$$

Solve graphically the following simultaneous equations, and compare by solving algebraically.

12. $2y-x=4$, $x+2y=3$.

13. $x+2y=5$, $2x-3y=6$.

14. $3x=5y$, $\frac{x}{2}+\frac{y}{3}=1$.

15. Draw the graph of the function $y=x^2-3x$ from $x=-2$ to $x=5$. With the same scale and axes, draw the graph of the function $y=\frac{1}{2}x$.

Solve graphically the simultaneous equations,

$$y=x^2-3x, \quad y=\tfrac{1}{2}x.$$

16. Draw the graph of the function $y=\frac{1}{10}x^3$ from $x=-2$ to $x=3$. With the same scale and axes, draw the graph of the function $y=\frac{1}{10}(4x+1)$.

For what values of x are these functions equal ? What equation in x can be solved in this way ?

17. The freezing point of water is 0° C. and 32° F. ; its boiling point is 100° C. and 212° F. Why is the graph for converting degrees Centigrade to degrees Fahrenheit a straight line ? Draw the graph for temperatures from $-40°$ C. to 100° C. [P.T.O.]

From the graph, (i) express in Fahrenheit, 35° C, 60° C., 73° C., $-10°$ C., $-30°$ C.; (ii) express in Centigrade, 98° F., 150° F., 180° F., 5° F.

What is the increase in degrees Fahrenheit if the temperature rises 10° C.? if the temperature rises $x°$ C.?

If F° Fahrenheit is the same temperature as C° Centigrade, express F as a function of C.

18. The marks obtained in an examination run from 23 to 87; draw a ready-reckoner graph for converting them so as to run from 0 to 100. Why is the graph a straight line?

Find from the graph (i) the scaled mark, if the original mark is 36, 50, 72; (ii) the original mark, if the scaled mark is 10, 44, 91; (iii) the mark which is unchanged by the conversion.

If the original mark is x and the scaled mark is y, express y as a function of x.

A boy, who does the paper late, obtains only 10 marks; how can you use the graph to find the corresponding scaled mark? Check from the equation between y and x.

19. Draw a graph which converts speeds given in miles per hour into feet per second, for speeds up to 60 miles an hour; 60 miles an hour $=88$ feet per second.

From the graph, (i) express in ft. per sec., 12, 26, 48, 53 m.p.h.
(ii) express in m.p.h., 20, 31, 50, 67 ft. per sec.

If x miles per hour is equivalent to y feet per second, express y as a function of x.

20. For a certain journey the charge for W lb. of luggage is P pence where $P=2\left(\frac{W}{5}-12\right)$. Draw a graph showing the scale of charges for weights from 75 lb. to 150 lb.

From the graph, find (i) the charge for 110 lb.; (ii) the weight for which the charge is half-a-crown.

How can you use the graph to find what luggage is allowed free? How much is it?

21. A man's rate of pay is as follows: ordinary time, 1s. 6d. an hour; overtime 2s. an hour. The ordinary working day is 7 hours. If he receives P shillings for working t hours one day, express P as a function of t (i) if $t<7$, (ii) if $t>7$.

Represent graphically the wages for one day's work, for values of t from 0 to 10. What is the meaning of the change of slope? How long does he work if he receives 14s. for one day's work?

[*For additional examples, see Appendix, Ex. S.* 10, *p.* 298.]

Graphical methods of solving equations are discussed in greater detail in Ch. XII and XV.

Constants in Formulae

Hooke's Law. If a spiral spring is suspended from one end, and if various loads are attached in turn to the other end, experiment shows that the *extension is proportional to the load*, unless the load is so heavy that the spring becomes injured. This is the principle of the spring-balance.

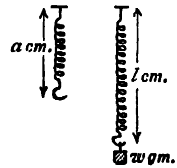


FIG. 166.

Let the natural length be a cm., and suppose that the length increases by b cm. for each 1 gm. weight attached, then the *total length*, l cm., of the spring when the load is w gm. is $(a+bw)$ cm.

$$\therefore \ \mathbf{l = bw + a}.$$

Here b, a are "*constants*" in the formula for l in terms of w, because their values remain the same when the load is changed. They do not of course remain the same for different springs, since a depends on the natural length and b on the elasticity of the spring.

Example 5. Show that the following observations for a spiral spring obey the law, $l = bw + a$, where a, b are constants, and find their values.

Load in gm., w	100	200	400	500
Total length in cm., l	52·6	53·4	55·0	55·8

By plotting these observations, we obtain the 4 points marked in Fig. 167, and we see that these points lie on a straight line;

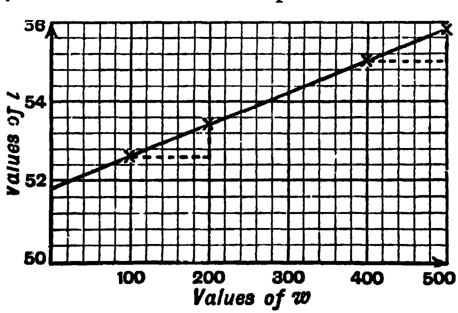


FIG. 167.

$\therefore$ l is a linear function of w, that is, $l = bw + a$ where a and b are constants.

The values of a, b can be obtained *either* from the graph *or* by calculation.

(i) *From the graph.* The line crosses tho l-axis where $l = 51 \cdot 8$, $\therefore$ $l = 51 \cdot 8$ when $w = 0$; $\therefore$ $a = 51 \cdot 8$.

Also the slope of the graph is upwards and is such that l increases by $0 \cdot 8$ when w increases by 100; $\therefore$ l increases by $0 \cdot 008$ for each unit increase in w; $\therefore$ $b = 0 \cdot 008$.

(ii) *By calculation.* From the table of values, when $w = 100$, $l = 52 \cdot 6$; $\therefore$ $52 \cdot 6 = 100b + a$.

Also when $w = 500$, $l = 55 \cdot 8$; $\therefore$ $55 \cdot 8 = 500b + a$.

Thus a, b are found by solving the simultaneous equations.

$$100b + a = 52 \cdot 6; \quad 500b + a = 55 \cdot 8.$$

These lead to the same values as before.

Hence the law for this spiral spring is

$$\mathbf{l = 0 \cdot 008w + 51 \cdot 8.}$$

EXERCISE X. d

Solve Numbers 1-5 *graphically and by calculation.*

1. The stretched length l in. of a spiral spring supporting a load of w ounces is observed to be as follows:

Load in oz., w -	8	12	24	30
Length in in., l -	22·4	23·6	27·2	29·0

Show that these observations obey the law, $l = a + bw$, where a, b are constants and find their values. What is the natural length of the spring? What load will stretch it to a length of 26·3 inches?

2. Water runs out of a tank so that after t minutes there are g gallons left in the tank, where $g = a - bt$, a and b being constants. After 5 minutes and 8 minutes, the number of gallons left is 700 and 520, respectively. Find the values of a, b. How much water was in the tank at the beginning? How long does the tank take to empty?

3. The following results were obtained with a screw-jack:

Load in lb., W - - -	10	20	30	50	75
Effort in lb., P - -	3·3	4·5	5·7	8·1	11·1

Show that these observations obey the law, $P = bW + c$, where b, c are constants. Find the values of b, c.

4. If a man's income is £x, for $160 < x < 400$, he pays a tax of £T, given by $T = bx + c$, where b, c are constants. On incomes of £300, £360, the tax is £22. 10s., £30 respectively; find the tax paid on an income of £200. Find also the values of b, c.

5. After n years' service, a man's salary, £S a year, is given by $S = a + bn$, where a, b are constants. After 4 years it is £200 a year, and after 12 years it is £320 a year. Find his salary after 6 years' service, also his salary for the first year. Find the values of a, b.

Solve Numbers 6-10 *by calculation only.*

6. If a polygon has n sides, the sum of its angles is $(a + bn)$ right angles where a, b are constants. What is the sum of the angles of a figure with (i) 3 sides, (ii) 4 sides? Hence find a, b.

7. If C° Centigrade is the same temperature as F° Fahrenheit, $C = a \cdot F + b$, where a, b are constants. Also 0° Centigrade = 32° Fahrenheit, and 100° C. = 212° Fahrenheit. Hence find the values of a, b and express 0° Fahrenheit in Centigrade, and 200° Centigrade as Fahrenheit.

8. Marks which run from 26 to 71 are to be scaled so as to run from 0 to 100; an original mark n becomes a scaled mark N, where $N = p + q \cdot n$ and p, q are constants. Find the values of p, q. Find also (i) the scaled mark corresponding to an original mark of 53; (ii) the mark which is unaltered by scaling.

9. If a polygon has n sides, it has $(bn + cn^2)$ diagonals where b, c are constants. How many diagonals has (i) a quadrilateral, (ii) a triangle? Use these answers to find the values of b, c. Test the formula by considering a 5-sided figure.

10. A framework is formed by rods jointed together at their ends. In order that the framework may be "just rigid" (*not over-stiff*), the number of joints j, and the number of rods r, must be connected by the formula, $r = a + bj$ where a, b are constants. If $j = 4$, $r = 5$, and if $j = 5$, then $r = 7$. What do these statements mean, and can you see why they are true? Use them to find the values of a, b. How many rods are required to make a framework just rigid, if there are 10 joints?
How many measurements must be made to copy accurately a polygon with j corners?

[*For a revision exercise on Ch. I-X, see Appendix, Ex. R.* 5, *p.* 265.]

CHAPTER XI

PRODUCTS, QUOTIENTS, AND FACTORS

Single Term Factors

If there is a factor common to each term of an expression, use **short division** to factorise the expression.

Example 1. Factorise $6a^2 - 12ax + 3a$.

$$\begin{array}{r|l} 3a & 6a^2 - 12ax + 3a \\ \hline & 2a \;-\; 4x \;+ 1 \end{array}$$

$$\therefore\ 6a^2 - 12ax + 3a = \mathbf{3a(2a - 4x + 1)}.$$

Example 2. Simplify $\dfrac{6a^2 - 12ax + 3a}{2a - 4x + 1}$.

The expression $= \dfrac{3a(2a - 4x + 1)}{2a - 4x + 1} = \mathbf{3a}$.

Example 3. Simplify $\dfrac{4x^3 + 12x^2y}{6x^2y - 18xy^2}$.

The expression $= \dfrac{4x^2(x + 3y)}{6xy(x - 3y)}$.

Divide numerator and denominator by $2x$,

$\therefore$ the expression $= \dfrac{\mathbf{2x(x + 3y)}}{\mathbf{3y(x - 3y)}}$.

Note. Leave the answer *in factors*, whenever possible.

EXERCISE XI. a

Simplify the following :

1. $(6c + 3d) \div 3$. **2.** $(6t^2 + 4t) \div 2t$. **3.** $(y^2 + y) \div y$.

4. $(8a^2 - 10a) \div 2a$. **5.** $(nr + r) \div r$. **6.** $(t^2 - t) \div t$.

7. $(x^2y - xy^2) \div xy$. **8.** $(n^4 - n^6) \div n^2$. **9.** $(9a^9 - 6a^6) \div 3a^3$.

Express, *where possible*, the following in factors ; *if there are no factors, say so.*

10. $5a + 10b$. **11.** $6c - 9$. **12.** $4n + 4$.

13. $xy + xz$. **14.** $p^2 - 2q$. **15.** $6c^2 - 4cd$.

16. $10r^2 + 10r$. **17.** $2 + 2s^2$. **18.** $b^2c - 2bc^2$.

19. $ab + a$. 20. $ab + b$. 21. $c^3 - 3$.
22. $3d^3 - d$. 23. $x^2 + y^2$. 24. $x^3y + y^3x$.
25. $9pq - 21rs$. 26. $x^2 - ax^2$. 27. $4yz^2 - 4$.
28. $ap + aq + ar$. 29. $x^3 - 3x^2 - 3x$. 30. $ax + ay + xy$.
31. $3r^2 - r(r + s)$. 32. $a^2 + ab + b^2$. 33. $c^2 + cd + c$.
34. $a(r + s) + b(r + s)$. 35. $c(x + y) - d(x - y)$.
36. $(p + q)^2 + a(p + q)$. 37. $a(b + c) + c(a + b)$.
38. $r(r + s) + s(s + r)$. 39. $(y - z)^2 + (y - z)$.

Simplify, *where possible*, the following expressions; *if there is no simpler form, say so.*

40. $\dfrac{2a + 2b}{2c}$. 41. $\dfrac{2r + 2s}{r + s}$. 42. $\dfrac{x^2 - xy}{2x}$.

43. $\dfrac{b^2 + bc}{ab}$. 44. $\dfrac{c + d}{d}$. 45. $\dfrac{c^2 - cd}{cd}$.

46. $\dfrac{3x - 3y}{x - y}$. 47. $\dfrac{t^2 + t}{t}$. 48. $\dfrac{4p + 6q}{6}$.

49. $\dfrac{at - bt}{ar - br}$. 50. $\dfrac{at^2 + bt}{ar^2 + br}$. 51. $\dfrac{4s^2}{6s^2 + 4s}$.

52. $\dfrac{cx + c}{c}$. 53. $\dfrac{y}{ay - by}$. 54. $\dfrac{c + d}{cz + dz}$.

55. $\dfrac{x - y}{x + y}$. 56. $\dfrac{n^5 + n^3}{n^4}$. 57. $\dfrac{1 + t}{t^2 + t}$.

58. $\dfrac{a^2 + a}{ab + b}$. 59. $\dfrac{x^2 + xy}{y^2 + xy}$. 60. $\dfrac{c^2 + c}{d^2 + d}$.

61. $\dfrac{p^2q - pq^2}{pqr}$. 62. $\dfrac{a - ac}{b - bc}$. 63. $\dfrac{xy}{xyz + z}$.

Relation between $\mathbf{x - y}$ and $\mathbf{y - x}$

Statements such as $4 + 6 = 6 + 4$, $2 + 7 = 7 + 2$, $8 + 5 = 5 + 8$, etc., are all included in the single formula, $x + y = y + x$.

Since $6 - 4 = 2$ and $4 - 6 = -2$, $\therefore (6 - 4) = -(4 - 6)$.

Similarly $2 - 7 = -(7 - 2)$, $8 - 5 = -(5 - 8)$, etc.; and all these statements are included in the single formula,

$$\mathbf{x - y = -(y - x)}.$$

This equality follows from the ordinary rules for removing brackets, $-(y - x) = -y + x = x - y$;

or from short division, $\begin{array}{r|l} -1 \,) & \underline{x - y} \\ & \underline{-x + y} \end{array}$.

Example 4. Divide $2a^2 - 2ab$ by $-2a$.

$$-2a\,\underline{)\,2a^2 - 2ab}\quad \underline{-a+b}.$$

Quotient, $b - a$.

Example 5. Simplify $\dfrac{5r - 5s}{6s - 6r}$.

$$\frac{5r-5s}{6s-6r} = \frac{5(r-s)}{6(s-r)} = \frac{-5(s-r)}{6(s-r)} = -\frac{5}{6}.$$

EXERCISE XI. b

1. If $a = 7$, $b = 4$, write down the values of :

(i) $a - b$; (ii) $b - a$; (iii) $(a-b)^2$; (iv) $(b-a)^2$;

(v) $(a-b)(b-a)$; (vi) $\dfrac{1}{a^2 - b^2}$; (vii) $\dfrac{1}{b^2 - a^2}$; (viii) $(b-a)^3$.

2. If $a + b = 9$, $a - b = 5$, write down the values of :

(i) $b + a$; (ii) $b - a$; (iii) $(b-a)^2$; (iv) $\dfrac{1}{b-a}$.

3. If $x - y = 4$, write down, where possible, the *numerical* values of :

(i) $y - x$; (ii) $\dfrac{1}{y+x}$; (iii) $(y-x)(x-y)$;

(iv) $3y - 3x$; (v) $1 - x - y$; (vi) $2 - x + y$.

4. If $r - s = 5$, find the values of :

(i) $\dfrac{2}{r-s} + \dfrac{1}{s-r}$; (ii) $\dfrac{4}{r-s} - \dfrac{1}{s-r}$;

(iii) $\dfrac{r-s}{s-r}$; (iv) $\dfrac{1-r+s}{1-s+r}$.

Simplify the following :

5. $a + (y - x)$. **6.** $a - (x - y)$. **7.** $a - (y - x)$.

8. $(p+q)(q+p)$. **9.** $(p-q)(q-p)$. **10.** $(p-q)^2 \div (q-p)$.

11. $\dfrac{-1}{z-y}$. **12.** $\dfrac{2(a-b)}{b-a}$. **13.** $\dfrac{4(b+a)}{6(a-b)}$.

14. $\dfrac{c-d}{2d-2c}$. **15.** $\dfrac{(y-z)^2}{(z-y)^2}$. **16.** $\dfrac{(c+d)(c-d)}{(d+c)(d-c)}$.

If $r - s = a$ and $y - z = b$, write down, *where possible*, in terms of a, b the following :

17. $1 + s - r$. **18.** $1 - y - z$. **19.** $(s-r)(z-y)$.

20. $r^2 - s^2$. **21.** $(r+y) - (s+z)$. **22.** $(r+z) - (y+s)$.

23. $(s-r)^2$. **24.** $r^2 + s^2$. **25.** $(s+r)(z+y)$.

Products

Geometrical Illustrations. Fig. 168 represents a rectangle, $(a+b)$ inches long, $(c+d)$ inches wide, divided into 4 compartments. The area of the rectangle is $(a+b)(c+d)$ sq. inches, and the areas of the compartments are shown in the figure.

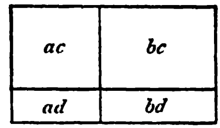


FIG. 168.

$$\therefore \ (\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) = \mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd}.$$

Thus $(3+5)(4+2) = 8 \times 6 = 48$

and $3 \times 4 + 3 \times 2 + 5 \times 4 + 5 \times 2 = 12 + 6 + 20 + 10 = 48$;

now draw the figure which illustrates this equality; see also Ch. VI, p. 87.

EXERCISE XI. c

1. What relation is illustrated by Fig. 169?

2. Illustrate by a rough figure that $(2a)^2 = 4a^2$.

3. What relation is illustrated by Fig. 170?

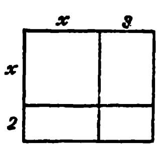


FIG. 169.

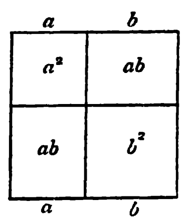


FIG. 170.

4. By drawing a rough figure, find the expanded value of $(y+7)(y+4)$.

5. What are the areas of the various compartments in Fig. 171?

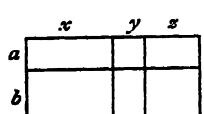


FIG. 171.

What relation does this figure illustrate?

6. Draw a rough figure whose area represents $(a+b+c)^2$, and use it to expand this expression.

7. Fig. 172 shows a square of side a inches, from which a

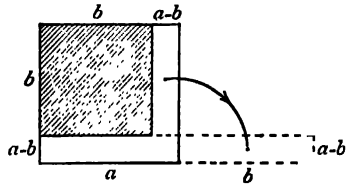


FIG. 172.

square of side b inches has been removed. Use this figure to express $a^2 - b^2$ as a product.

Products by Inspection

$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$, as before.

To multiply $(a+b)$ *by* $(c+d)$, *we multiply each of the terms in* $(a+b)$ *by each of the terms in* $(c+d)$ *and then add the separate products.*

Example 6. Expand $(p-q)(r-s)$.

$$(p-q)(r-s) = p(r-s) - q(r-s) = \mathbf{pr - ps - qr + qs}.$$

The middle step should be performed mentally; say to yourself, $p \cdot r + p(-s) - q \cdot r - q(-s)$ and write down merely,

$$pr - ps - qr + qs.$$

Example 7. Expand $(3x-2)(4x+1)$.

$$(3x-2)(4x+1) = 12x^2 + 3x - 8x - 2 = \mathbf{12x^2 - 5x - 2}.$$

After a little practice, *the middle step should be performed mentally.* The insertion of links, as shown below, helps mental calculation.

$$(3x - \overleftrightarrow{2)(4x} + 1).$$

For longer expressions, the working may be arranged as in Arithmetic.

Example 8. Multiply $2x^2 - 5x - 3$ by $3x + 7$.

$$\begin{array}{llll}
2x^2 - & 5x & - 3 & \\
3x + & 7 & & \\
\hline
6x^3 - & 15x^2 - & 9x & \\
 & 14x^2 - & 35x - & 21 \\
\hline
6x^3 - & x^2 - & 44x - & 21 \\
\hline
\end{array}$$

But even in such cases as this, it is useful to practise collecting coefficients of like terms mentally.

EXERCISE XI. d

Expand the following expressions :

1. $(a+b)(x+y)$.
2. $(a+b)(x-y)$.
3. $(a-b)(x+y)$.
4. $(a-b)(x-y)$.
5. $(a+b)(a-c)$.
6. $(b-x)(b+y)$.
7. $(c-z)(d-z)$.
8. $(p-r)(q+r)$.
9. $(r+s)(s+t)$.
10. $(a+z)(b+3)$.
11. $(c-1)(d+2)$.
12. $(p-2)(q-2)$.
13. $(x+2)(x+3)$.
14. $(y+4)(y+7)$.
15. $(z+1)(z+10)$.
16. $(a+5)(a-2)$.
17. $(b-7)(b+3)$.
18. $(c-3)(c-5)$.
19. $(p-1)(p-1)$.
20. $(q+1)(q-1)$.
21. $(4+t)(3-t)$.
22. $(2-s)(5-s)$.
23. $(1-n)(5+n)$.
24. $(3-x)(3+x)$.
25. $(2x+1)(3x+1)$.
26. $(4y-1)(3y+1)$.
27. $(5z-1)(5z+1)$.
28. $(2a+1)(a+4)$.
29. $(3b+2)(4b+5)$.
30. $(3c-2)(4c-5)$.
31. $(3d+2)(4d-5)$.
32. $(2n-7)(3n-1)$.
33. $(4p-3)(3p+2)$.
34. $(a+3b)(a+5b)$.
35. $(x-2y)(x+4y)$.
36. $(2c-d)(3c-d)$.
37. $(a+b)^2$.
38. $(a-b)^2$.
39. $(a+b)(a-b)$.
40. $(x-3y)^2$.
41. $(x+5y)^2$.
42. $(x+4y)(x-4y)$.
43. $(3a+b)^2$.
44. $(3c-4d)^2$.
45. $(5r+3s)(5r-3s)$.
46. $(3y+z)(z-3y)$.
47. $(a+\frac{1}{2}b)^2$.
48. $(\frac{1}{2}x-y)^2$.
49. $(x^2+x+2)(x+3)$.
50. $(y+1)(y^2-2y+3)$.
51. $(a^2+2ab+3b^2)(a+b)$.
52. $(y+z)(y^2-yz+z^2)$.
53. $(2c^2-3c-1)(3c-2)$.
54. $(3r^2-rs-2s^2)(2r+3s)$.
55. $(2+x)(1-7x-3x^2)$.
56. $(2-y-5y^2)(3-2y)$.
57. $(x^2+2xy+4y^2)(x-2y)$.
58. $(4z-3)(5-3z-2z^2)$.

Find the coefficient of x^2 and the coefficient of x in the following products:

59. $(x-1)(5x^2-4x-3)$.

60. $(1-2x)(3+7x-2x^2)$.

61. $(3x+5)(2x^2-x-2)$.

62. $(4-5x-8x^2)(4+5x)$.

63. $(4x^2+3x-7)(4x-3)$.

64. $(2x-3-5x^2)(3-2x)$.

Solve the following equations:

65. $(x+1)(x-2)+5=(1-x)(2-x)$.

66. $3(n-1)(n-2)=(1-n)(1-3n)$.

67. $(y+1)^2+y^2+1=(y-1)^2+y^2-1$.

68. $(t+1)(2-t)+(3+t)(2t+1)+(2+t)(3-t)=0$.

69. $(z-1)(z+3)(z+2)+(z+1)(z-3)(z+2)=2(z+1)(z+3)(z-2)$.

70. $(y+2)(y+3)(y+5)=(y-5)(y-4)(y+3)+(y-3)(16y+5)$.

[*Note. For additional drill-examples, see Exercise E.P.* 16, *p.* 251.]

Important Expansions

Three of the expansions in the last exercise should be committed to memory:

$$(\mathbf{A}+\mathbf{B})^2 \equiv \mathbf{A}^2+\mathbf{B}^2+2\mathbf{AB}$$

$$(\mathbf{A}-\mathbf{B})^2 \equiv \mathbf{A}^2+\mathbf{B}^2-2\mathbf{AB}$$

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) \equiv \mathbf{A}^2-\mathbf{B}^2$$

In words,

The square of the sum of two numbers is equal to the sum of their squares PLUS *twice their product.*

The square of the difference of two numbers is equal to the sum of their squares MINUS *twice their product.*

The product of the sum and the difference of two numbers is equal to the difference of their squares.

Perfect Squares

Example 9. Write down the squares of $2x+3y$ and $5x-7y$.

$$(2x+3y)^2=(2x)^2+(3y)^2+2(2x)(3y)=\mathbf{4x^2+9y^2+12xy}.$$

$$(5x-7y)^2=(5x)^2+(7y)^2-2(5x)(7y)=\mathbf{25x^2+49y^2-70xy}.$$

After a little practice, *the middle step should be performed mentally*, not written down.

Example 10. Is $16y^2 + 25z^2 - 20yz$ a perfect square ?

If it is a perfect square, it must be the square of $4y - 5z$.

Now $$(4y - 5z)^2 = 16y^2 + 25z^2 - 40yz.$$

$\therefore$ $16y^2 + 25z^2 - 20yz$ is not a perfect square.

Square Roots. Any positive number has two square roots; thus the square roots of 25 are $+5$ and -5.

Since $(\mathrm{B}-\mathrm{A})^2 = \mathrm{B}^2 + \mathrm{A}^2 - 2\mathrm{BA} = (\mathrm{A}-\mathrm{B})^2$, it follows that $\mathrm{A}^2 + \mathrm{B}^2 - 2\mathrm{AB}$ has two square roots, namely $\mathrm{A}-\mathrm{B}$ and $\mathrm{B}-\mathrm{A}$; these may be written $\mathrm{A}-\mathrm{B}$ and $-(\mathrm{A}-\mathrm{B})$ or, more shortly, $\pm(\mathrm{A}-\mathrm{B})$.

Similarly, $(-\mathrm{A}-\mathrm{B})^2 = \mathrm{A}^2 + \mathrm{B}^2 + 2\mathrm{AB} = (\mathrm{A}+\mathrm{B})^2$; therefore the two square roots of $\mathrm{A}^2 + \mathrm{B}^2 + 2\mathrm{AB}$ are $\mathrm{A}+\mathrm{B}$ and $-\mathrm{A}-\mathrm{B}$; these may be written $\mathrm{A}+\mathrm{B}$ and $-(\mathrm{A}+\mathrm{B})$ or, more shortly, $\pm(\mathrm{A}+\mathrm{B})$.

Completing the Square

Geometrical Illustration. What must be added to $x^2 + 6x$ to make the result a perfect square ?

Fig. 173 represents a rectangle $(x+6)$ in. long, x in. high; $\therefore$ its area $= x(x+6)$ sq. in. $= (x^2 + 6x)$ sq. in.

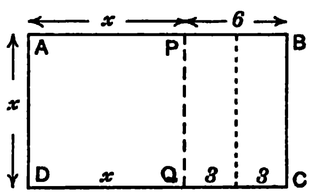


Fig. 173.

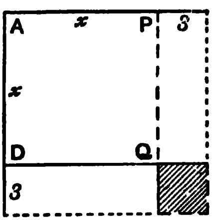


Fig. 174.

Transpose half the rectangle PBCQ and fit it on to DQ, as in Fig. 174. This gives a square of side x in., bordered with two rectangles, each of width $\frac{6}{2}$ in. $= 3$ in. To complete the square, add the shaded area in Fig. 174; this is a square of side 3 in., area 3^2 sq. in. $= 9$ sq. in.

The result is a square of side $(x+3)$ in., area $(x+3)^2$ sq. in. $= (x^2 + 6x + 9)$ sq. in.

Hence, to $x^2 + 6x$, add $(\frac{6}{2})^2 = 9$; then the sum is $(x+3)^2$.

To complete the square, start by looking for an expression whose square is of the required form.

Example 11. What must be added to $a^2 - 5a$ to make the sum a perfect square? Of what expression is the sum the square?

Since $$a^2 - 2ab + b^2 = (a-b)^2,$$
$$a^2 - 5a + \ ? = a^2 - 2(\tfrac{5}{2})a + \ ? = (a - \tfrac{5}{2})^2$$
$$= a^2 - 5a + (\tfrac{5}{2})^2.$$

$\therefore$ if $(\frac{5}{2})^2$ is added to $a^2 - 5a$, the sum equals $(a - \frac{5}{2})^2$, and this is the square of $a - \frac{5}{2}$, also of $-(a - \frac{5}{2})$, that is, $\frac{5}{2} - a$.

Example 12. What must be added to $25x^2 + 7x$ to make the sum a perfect square?

$$25x^2 + 7x + \ ? = (5x)^2 + 2(5x)(\tfrac{7}{10}) + \ ? = (5x + \tfrac{7}{10})^2$$
$$= 25x^2 + 7x + (\tfrac{7}{10})^2.$$

$\therefore$ if $(\frac{7}{10})^2$ is added to $25x^2 + 7x$, the sum equals $(5x + \frac{7}{10})^2$.

The Difference of Two Squares

Since $$\mathsf{A}^2 - \mathsf{B}^2 = (\mathsf{A} + \mathsf{B})(\mathsf{A} - \mathsf{B}),$$
$$x^2 - 9 = x^2 - 3^2 = (x+3)(x-3),$$
and $$49a^2 - 25b^2 = (7a)^2 - (5b)^2 = (7a + 5b)(7a - 5b),$$
and $$(x+p)^2 - q^2 = [(x+p) + q][(x+p) - q]$$
$$= (x + p + q)(x + p - q).$$

Example 13. Factorise $16a^2 - 9(b - c)^2$.

$$16a^2 - 9(b-c)^2 = [4a]^2 - [3(b-c)]^2$$
$$= [4a + 3(b-c)][4a - 3(b-c)]$$
$$= (4a + 3b - 3c)(4a - 3b + 3c).$$

EXERCISE XI. e

Write down the squares of the following:

1. $a + 5$. 2. $b - 3$. 3. $3 - b$. 4. $-a - 5$

5. $x - y$. 6. $y - x$. 7. $4c + d$. 8. $4c - d$.

9. $2p + 3q$. 10. $-2p - 3q$. 11. $5q - 7p$. 12. $7p - 5q$

13. $4y + \frac{1}{2}$. 14. $z - \frac{1}{2}$. 15. $a + \dfrac{1}{b}$. 16. $c - \dfrac{3}{d}$.

17. $a + \dfrac{1}{a}$. 18. $b - \dfrac{1}{b}$. 19. $3ab - 4c$. 20. $x + \dfrac{3y}{2}$.

Expand the following :

21. $(a+3b)(a-3b)$.
22. $(4x+5y)(4x-5y)$.
23. $(4c-5d)(4c-5d)$.
24. $(1-7t)(1+7t)$.
25. $(1-x^3)(1+x^3)$.
26. $(a-b)(b-a)$.
27. $(2p+\frac{1}{2}q)(2p-\frac{1}{2}q)$.
28. $\left(c-\dfrac{1}{c}\right)\left(c+\dfrac{1}{c}\right)$.
29. $(x+2y)(2x-y)$.

In Nos. 30-44, state whether the expression is a perfect square ; if it is, give two square roots.

30. x^2-2x+1.
31. $x^2+2xy+y^2$.
32. a^2-6a+9.
33. b^2+2b+4.
34. c^2-4c+4.
35. $d^2-10d-25$.
36. $4-4k+k^2$.
37. n^2-25.
38. $r^2+r+\frac{1}{4}$.
39. $100+60s+9s^2$.
40. $x^2-6xy+36y^2$.
41. $1-2p+p^2$.
42. $s^2+\dfrac{1}{s^2}+2$.
43. $4a^2-20ab+25b^2$.
44. $x^2-40x+400$.

What term, added to the following, makes the sum a perfect square ? Of what expression is the sum the square ?

45. x^2+10x.
46. y^2-8y.
47. y^2+2yz.
48. b^2-2bc.
49. z^2-4z.
50. a^2+b^2.
51. $2rs+s^2$.
52. t^2+16.
53. a^2+3a.
54. c^2-7c.
55. b^2+9.
56. $9d^2-30d$.
57. n^2-n.
58. a^2+ab.
59. $9y^2+16z^2$.
60. $1+25c^2$.

Factorise the following :

61. y^2-z^2.
62. a^2+6a+9.
63. $9-x^2$.
64. $x^2-8x+16$.
65. b^2-16c^2.
66. $t^2+14t+49$.
67. $4n^2-25$.
68. $p^2-10pq+25q^2$.
69. $1-y^2$.
70. $1+14b+49b^2$.
71. $a^2b^2-c^2$.
72. $a^2b^2-2abc+c^2$.
73. $4p^2-25q^2$.
74. $1-4t^2$.
75. ab^2-ac^2.
76. $3s^2-12$.
77. $5-45n^2$.
78. $(x+3)^2-4$.
79. $(y-4)^2-25$.
80. $(x+2y)^2-z^2$.
81. $(b-c)^2-16$.
82. $1-(a-b)^2$.
83. $x^2-(y+z)^2$.
84. $4a^2-9(b+c)^2$.
85. a^4-b^4.
86. $p^2-16(q-r)^2$.
87. $x^4-4x^2y^2$.

88. Use factors to evaluate (i) 25^2-24^2; (ii) 46^2-44^2; (iii) 97^2-87^2; (iv) $5{\cdot}4^2-4{\cdot}6^2$.

89. Evaluate R^2-r^2 when $R=3\frac{1}{4}$ and $r=2\frac{3}{4}$.

90. What is the quotient if (i) b^2-c^2 is divided by $b-c$; (ii) p^2-q^2 is divided by $p+q$; (iii) a^2-9b^2 is divided by $a-3b$; (iv) $4x^2-25y^2$ is divided by $2x+5y$?

Factors by Grouping Terms

When factorising an expression, first see if there is a common factor of each term. If so, write it down, and find the other factor by short division.

Example 14. Factorise $p(a+b)+q(a+b)$.

$(a+b)$ is a factor of each term.

$$[a+b]\ \underline{)\ p(a+b)+q(a+b)}$$
$$\underline{p\quad+\quad q}$$

$$\therefore\ p(a+b)+q(a+b)=(\mathbf{a}+\mathbf{b})(\mathbf{p}+\mathbf{q}).$$

The short division should be done mentally.

Sometimes a common factor can be found by grouping terms together.

Example 15. Factorise $ax-ay+bx-by$.

$$ax-ay+bx-by=a(x-y)+b(x-y).$$

Here $(x-y)$ is a factor of each term ;

$\therefore$ by short division, $ax-ay+bx-by=(\mathbf{x}-\mathbf{y})(\mathbf{a}+\mathbf{b})$.

When dividing by a common factor, write it down first, and treat it as the divisor in a division sum.

When you have obtained the factors, multiply them together mentally, to make sure that their product equals the given expression.

The fact that an expression can be arranged in two groups does not mean, necessarily, that it can be factorised.

Thus, $\quad ac+ax+bc+by=a(c+x)+b(c+y)$.

But there is no factor common to these two terms, so we cannot use the "short division" method.

Actually, this expression has no factors.

An expression, such as $a(c+x)+b(c+y)$, which is written as the *Sum* of two terms is not in factors.

Thus, $29=15+14=3\times 5+2\times 7$, but these numbers are not factors of 29.

Example 16. Factorise $a^2+bc+ab+ac$.

$$a^2+bc+ab+ac=(a^2+bc)+a(b+c).$$

This is not in factors because it is the *sum* of two terms ; also there is no factor common to the two terms, so we cannot use the "short division" method.

But $\quad a^2+bc+ab+ac=a^2+ab+bc+ac$
$$=a(a+b)+c(b+a).$$
Here, $(a+b)$ is a factor of each term, since $a+b=b+a$;

$\therefore$ by short division, $a^2+bc+ab+ac=(\mathbf{a}+\mathbf{b})(\mathbf{a}+\mathbf{c})$.

Note. *Group together terms which have a common factor.*

Example 17. Factorise $ad+bc-ac-bd$.
$$ad+bc-ac-bd=ad-ac+bc-bd=a(d-c)+b(c-d)$$
$$=a(d-c)-b(d-c), \text{ since } c-d=-(d-c),$$
$$=(\mathbf{d}-\mathbf{c})(\mathbf{a}-\mathbf{b}).$$

EXERCISE XI. f

[*In this exercise, Nos.* 1-25 *are intended for oral discussion.*]

1. Is $a+1$ a factor of (i) $2a+2$; (ii) $3+3a$; (iii) $a+2$?
2. Is $b+1$ a factor of (i) $2+2b$; (ii) $-b-1$; (iii) $(1+b)^2$?
3. Is $c-1$ a factor of (i) $2c-1$; (ii) $1-c$; (iii) $c+1$?
4. Is $x-3$ a factor of (i) $x+3$; (ii) $3-x$; (iii) x^2-3?
5. Is $x+y$ a factor of (i) $2y+2x$; (ii) $ay+ax$; (iii) $-y+x$?
6. Is $x-y$ a factor of (i) $3y-3x$; (ii) $ax-by$; (iii) x^2-y^2?
7. Is $a-b$ a factor of (i) $4(b-a)$; (ii) $-a-b$; (iii) a^2-b^2?
8. Is $c+d$ a factor of (i) cd; (ii) c^2+d^2; (iii) c^2-d^2?
9. Is $a+b$ a factor of $x(a+b)+y(b+a)$?
10. Is $c+d$ a factor of $a(c+d)-b(c-d)$?
11. Is $x+y$ a factor of $x(a+b)+y(a+c)$?
12. Is $a-b$ a factor of $n(a-b)+a-b$?
13. Is $c-d$ a factor of $n(c-d)-c-d$?
14. Is $x-y$ a factor of $a(y-x)+b(x-y)$?
15. Is $x-y$ a factor of $x(a+b)-y(a+c)$?

Have the following expressions factors? If so, find them and check by multiplication. *If there are no factors, say so.*

16. $a(c+d)-b(c+d)$.
17. $p(x+y)-q(x-y)$.
18. $a(x+y)+b(x+z)$.
19. $a(c+d)-b(d+c)$.
20. $c(a-b)+d(b-a)$.
21. $x(2y+2)+z(y+1)$.
22. $a(b+1)+b(a+1)$.
23. $x-a+b(a-x)$.
24. $a(x+y)+x+y$.
25. $a(x+1)-b(1-x)$.

Factorise, *when possible*, the following expressions. *If there are no factors, say so.*

26. $a^2+ab+ac+bc$.

27. $a^2-ab-ac-bc$.

28. $ax-ay+bx-by$.

29. $ac+ad-bc-bd$.

30. $ax+ay+bx+bz$.

31. $ap+aq+bq+bp$.

32. $ca-cd-bd+ba$.

33. $ax+ap-cx-pc$.

34. $x^2+xy+3x+3y$.

35. $a^2+ac-5a-5c$.

36. $y^2-yz-2y-2z$.

37. $x(a+b+c)+y(a+b+c)$.

38. $a^2c^2+a^2d^2+b^2d^2+b^2c^2$.

39. $2ab+2ac+b+c$.

40. $5cx+5dy-5cy-5dx$.

41. $a^2-ab-2a-2b$.

42. $xy+y^2-x-y$.

43. $a(x+y)+c(x-y)$.

44. $a^2(b+c)-bc(c+b)$.

45. $6ab-3bx+2ay-xy$.

46. $4x^2-2xy-6xz+3yz$.

47. x^3+x+x^2+1.

48. $x^2+(a+b)x+ab$.

49. $x^2-(c+d)x+cd$.

50. $2px+qy-py-2qx$.

51. $ab-12xy+3bx-4ay$.

52. $ax-3+a-3x$.

53. $ac+ad-a^2-cd$.

54. $x^2-x+y-xy$.

55. $x^2-2x-xy-2y$.

56. $4-4x+cx-c$.

57. $l(a-b)-m(a+b)$.

58. $l(a-b)+m(b-a)$.

59. $2a^4-2a^3x+x-a$.

60. $1+c^2+cd+c^3d$.

61. $ax-2ay+2bx-by$.

Quadratic Functions

The product of two first-degree functions of x is a quadratic function of x.

Thus, $(2x-5)(3x+4)=2x(3x+4)-5(3x+4)$

$$=6x^2+8x-15x-20=6x^2-7x-20.$$

To factorise a quadratic function, we express it so that this process can be *worked backwards*.

Example 18. Factorise $8x^2+10x+3$.

Replace $+10x$ by two terms whose product is $8x^2\times 3$.

$$8x^2\times 3=24x^2=4x\times 6x.$$

$$\therefore\ 8x^2+10x+3=8x^2+4x+6x+3$$
$$=4x(2x+1)+3(2x+1)=(2x+1)(4x+3).$$

Check the answer by multiplying mentally.

Example 19. Factorise $y^2 - 22y + 96$.

Replace $-22y$ by two terms whose product is $y^2 \times 96$.

$$96y^2 = 2y \times 48y = 3y \times 32y = 4y \times 24y = 6y \times 16y.$$

$$\therefore\ y^2 - 22y + 96 = y^2 - 6y - 16y + 96$$
$$= y(y-6) - 16(y-6) = (\mathbf{y-6})(\mathbf{y-16}).$$

Example 20. Factorise $6x^2 + 11xy - 10y^2$.

Replace $+11xy$ by two terms whose product is $6x^2 \times (-10y^2)$.

$$-60x^2y^2 = (-4xy) \times (+15xy).$$

$$\therefore\ 6x^2 + 11xy - 10y^2 = 6x^2 - 4xy + 15xy - 10y^2$$
$$= 2x(3x-2y) + 5y(3x-2y) = (\mathbf{3x-2y})(\mathbf{2x+5y}).$$

Example 21. Factorise $6 + 11t - 10t^2$.

Replace $+11t$ by two terms whose product is $6 \times (-10t^2)$.

$$-60t^2 = (-4t) \times (+15t).$$

$$\therefore\ 6 + 11t - 10t^2 = 6 - 4t + 15t - 10t^2$$
$$= 2(3-2t) + 5t(3-2t) = (\mathbf{3-2t})(\mathbf{2+5t}).$$

If the coefficient of t^2 is negative and if the constant term is positive, work, as in Example 21, in *ascending* powers of t. *Do not turn the expression round.*

Example 22. Factorise $12a^2 - a - 35$.

Replace $-a$ by two terms whose product is $12a^2 \times (-35)$.

$$(12a^2) \times (-35) = -2^2 \times 3 \times 5 \times 7 \times a^2 = (+20a) \times (-21a).$$

$$\therefore\ 12a^2 - a - 35 = 12a^2 + 20a - 21a - 35$$
$$= 4a(3a+5) - 7(3a+5) = (\mathbf{3a+5})(\mathbf{4a-7}).$$

If the product is *positive*, the numerical factors, we look for, have a known sum; if the product is *negative*, they have a known difference.

If the product has a large number of factors, as in Example 22, it is best to express it in prime factors before looking for the pair which have a known sum or difference.

EXERCISE XI. g

Write down the coefficient of x in Nos. 1-9.

1. $(2x+3)(x+4)$.
2. $(2x-3)(x-4)$.
3. $(2x+3)(x-4)$.
4. $(2x-3)(x+4)$.
5. $(3x-7)(2x+3)$.
6. $(x+1)(7x-5)$.
7. $(4x-1)(3x-2)$.
8. $(1-2x)(2+5x)$.
9. $(7-3x)(2-3x)$.

Find *by inspection* two numbers satisfying the conditions in Nos. 10-25.

10. Product $=15$, sum $=8$.

11. Product $=15$, sum $=16$.

12. Product $=12$, sum $=8$.

13. Product $=108$, sum $=24$.

14. Product $=45$, difference $=4$.

15. Product $=60$, difference $=11$.

16. Product $=48$, difference $=13$.

17. Product $=54$, difference $=3$.

18. Product $=24$, sum $=-11$.

19. Product $=36$, sum $=-15$.

20. Product $=-18$, sum $=7$.

21. Product $=-18$, sum $=-7$.

22. Product $=-28$, sum $=-3$.

23. Product $=-45$, sum $=12$

24. Product $=-40$, sum $=3$.

25. Product $=-96$, sum $=-10$.

Factorise the following: *check your answers mentally.*

26. $3x^2+5x+2$.
27. $2y^2+7y+3$.
28. $2z^2+11z+5$.

29. a^2+6a+8.
30. $b^2+7b+10$.
31. $c^2+10c+9$.

32. $r^2-7r+12$.
33. $s^2-5s-14$.
34. $t^2+6t-16$.

35. $3y^2+8y+4$.
36. $2z^2+5z-3$.
37. $2p^2-11p+12$.

38. $15+2s-s^2$.
39. $4-3t-t^2$.
40. $2-k-6k^2$.

41. $n^2+7n-18$.
42. $x^2-3x-28$.
43. $2y^2-22y+48$.

44. $3z^2+13z+4$.
45. $12a^2+33a-9$.
46. $3p^2-10p+8$.

47. $b^2+15b+14$.
48. $12+11c-c^2$.
49. $3a^2+60a-63$.

50. $9r^2-39r-30$.
51. $4p^2-13p+3$.
52. $6+5y-6y^2$.

53. $x^2+3x-70$.
54. $2y^2-16y-96$.
55. $c^2-13c+40$.

56. $8a^2-14a+3$.
57. $6n^2+5n-6$.
58. $60k^2-25k-10$.

59. $2a^2-22ab+60b^2$.
60. $x^2+7xy-30y^2$.
61. $r^2-8rs-84s^2$.

62. $4b^2+13bc+3c^2$.
63. $18-3x-x^2$.
64. $12x^2-11xy+2y^2$.

65. $12a^2-7ab+b^2$.
66. $40l^2-50lm-15m^2$.
67. $60+3y-3y^2$.

68. $27z^2+18z-24$.
69. $6a^2-13a+6$.
70. $4b^2+16b+15$.

71. $12c^2+7c-12$.
72. $24-14x-20x^2$.
73. $24+50y-25y^2$.

Factors by Inspection

After a little practice, *simple* quadratic functions can often be factorised at sight, without using the grouping method. **But always check the answer by (mental) multiplication.**

Example 23. Factorise (i) $x^2+11x+24$; (ii) $x^2-2x-35$.

(i) Find two numbers whose product is $+24$ and sum $+11$. These are $+8$, $+3$.

$$x^2+11x+24=(\mathbf{x}+8)(\mathbf{x}+3).$$

Check by mental multiplication.

(ii) Find two numbers whose product is -35 and sum -2. These are -7, $+5$.

$$x^2-2x-35=(\mathbf{x}-7)(\mathbf{x}+5).$$

Check as before.

Example 24. Factorise $2y^2+5y-3$.

Write down pairs of factors whose products introduce $2y^2$ and -3, and select that pair which also introduces $+5y$. Possible pairs are

$$(2y+3)(y-1)\,;\quad (2y-3)(y+1)\,;\quad (2y-1)(y+3)\,;\quad (2y+1)(y-3).$$

$$2y^2+5y-3=(2\mathbf{y}-1)(\mathbf{y}+3).$$

Use the grouping method, whenever you are not able to obtain the factors by inspection, *quickly.*

EXERCISE XI. h

Factorise the following: *check your answers mentally.*

1. a^2+5a+6.	2. b^2+6b+9.	3. $c^2+8c+12$.
4. $x^2-7x+12$.	5. $y^2-3y-10$.	6. z^2+2z-8.
7. $p^2-8p+16$.	8. q^2+4q-5.	9. r^2-r-6.
10. $s^2+9s+18$.	11. $t^2-3t-28$.	12. $n^2-11n+30$.
13. k^2+k-30.	14. $a^2+12a+36$.	15. $b^2-5b-50$.
16. $c^2-14c+49$.	17. $d^2+6d-27$.	18. $p^2+14p+45$.
19. $q^2+7q-60$.	20. $r^2-18r+72$.	21. $s^2-9s-70$.
22. t^2-36.	23. n^2+n-90.	24. $k^2-2k-63$.
25. $y^2+16y+64$.	26. a^2-64.	27. $z^2-16z+64$.
28. $b^2-7b-120$.	29. $c^2-20c+100$.	30. x^2-100.
31. $x^2-12xy+32y^2$.	32. $a^2+3ab-10b^2$.	33. $c^2-4cd-32d^2$.
34. $y^2+yz-6z^2$.	35. $b^2-4bc-5c^2$.	36. $x^2+8xy+16y^2$.
37. $x^2+6xy+9y^2$.	38. x^2-9y^2.	39. $x^2-6xy+9y^2$.
40. $1+3a-10a^2$.	41. $1-2b-24b^2$.	42. $1-12c+35c^2$.

43. $1 - 14x + 49x^2$.
44. $1 - 49y^2$.
45. $1 + 14z + 49z^2$.
46. $12 - a - a^2$.
47. $10 + 7c + c^2$.
48. $35 - 2d - d^2$.
49. $1 - y - 20y^2$.
50. $24 - 5z - z^2$.
51. $28 - 11t + t^2$.
52. $3a^2 - 12a - 15$.
53. $2b^2 + 2b - 12$.
54. $5n^2 - 5n - 150$.
55. $4c^2 - 100$.
56. $a^2b^2 - 7abc + 10c^2$.
57. $1 - 3xy - 18x^2y^2$.
58. $2y^2 + 6yz - 20z^2$.
59. $3 - 27n^2$.
60. $24r^2 + 2rs - s^2$.
61. $2a^2 - 5a + 2$.
62. $3b^2 + 5b - 2$.
63. $6c^2 - 11c + 3$.
64. $2p^2 - 7p - 15$.
65. $5q^2 - 16q + 3$.
66. $4d^2 + 5d - 6$.
67. $5y^2 + 14y - 3$.
68. $4z^2 - 11z + 6$.
69. $5a^2 + 3a - 2$.
70. $10b^2 - 13b + 4$.
71. $12c^2 - 11c - 5$.
72. $10k^2 - 21k + 9$.
73. $12n^2 + 19n + 5$.
74. $10x^2 + 3x - 4$.
75. $10y^2 + 9y - 1$.
76. $6p^2 + 7p - 10$.
77. $4x^2 - 12x + 9$.
78. $9a^2 + a - 10$.
79. $10y^2 - 43y + 12$.
80. $8b^2 + 7b - 15$.
81. $6c^2 - 17c + 10$.
82. $9a^2 - 64b^2$.
83. $25x^2 + 40xy + 16y^2$.
84. $42c^2 - cd - 30d^2$.

85. If $x + 2$ is a factor of $x^2 + ax + 10$, what is the other factor? Hence find a.

86. What is b if $x + 3$ is a factor of $x^2 + bx - 12$?

87. What is c if $x - 6$ is a factor of $x^2 + cx + 30$?

88. What is a if $x - 4$ is a factor of $x^2 + ax - 4$?

89. What is b if $x - 3$ is a factor of $x^2 + bx - 9$?

90 What is c if $x + 4$ is a factor of $x^2 + 7x + c$?

91 What is a if $x + 2$ is a factor of $x^2 - 5x + a$?

92. What is b if $2x - 3$ is a factor of $6x^2 + bx - 12$?

[*Note. For additional drill-examples, see Exercise E.P.* 17, *p.* 252. *For harder miscellaneous questions on factors, see Appendix, Ex. S.* 11, *p.* 301.]

Long Multiplication and Division

A simple example of long multiplication has been given on p. 177; the following section, dealing with both multiplication and division, may be omitted at a first reading, without affecting the work of the later chapters.

Example 25. Multiply $2x^3-4x^2+2x+3$ by $3x^2-x-5$.

$$\begin{array}{rrrrrr}
2x^3 & -\ 4x^2 & +\ 2x & +\ 3 & & \\
3x^2 & -\ x & -\ 5 & & & \\
\hline
6x^5 & -\ 12x^4 & +\ 6x^3 & +\ 9x^2 & & \\
 & -\ 2x^4 & +\ 4x^3 & -\ 2x^2 & -\ 3x & \\
 & & -\ 10x^3 & +\ 20x^2 & -\ 10x & -\ 15 \\
\hline
\mathbf{6x^5} & \mathbf{-\ 14x^4} & & \mathbf{+\ 27x^2} & \mathbf{-\ 13x} & \mathbf{-\ 15}
\end{array}$$

Example 26. Multiply a^2-ab+b^2 by a^2+ab+b^2.

$$\begin{array}{rrrrr}
a^2 & -ab & +b^2 & & \\
a^2 & +ab & +b^2 & & \\
\hline
a^4 & -a^3b & +a^2b^2 & & \\
 & +a^3b & -a^2b^2 & +ab^3 & \\
 & & +a^2b^2 & -ab^3 & +b^4 \\
\hline
\mathbf{a^4} & & \mathbf{+a^2b^2} & & \mathbf{+b^4}
\end{array}$$

Example 27. Divide $3x^3+5x^2-14x+4$ by $x+3$.

First Method : *Inverse Multiplication.*

Fill in the blank spaces in the following :

$$3x^3+5x^2-14x+4=(x+3)(\ldots\ldots\ldots)+\ldots\ldots\ldots$$

$(x+3)\,.\,3x^2=3x^3+9x^2$; but we want $5x^2$ instead of $9x^2$, so we must arrange for a term, $-4x^2$.

$(x+3)(3x^2-4x)=3x^3+5x^2-12x$; but we want $-14x$ instead of $-12x$, so we must arrange for a term, $-2x$.

$(x+3)(3x^2-4x-2)=3x^3+5x^2-14x-6$; but we want $+4$ instead of -6, so we add 10 to each side.

$\therefore\ (x+3)(3x^2-4x-2)+10=3x^3+5x^2-14x+4.$

$\therefore$ the quotient is $\mathbf{3x^2-4x-2}$, and the remainder is **10.**

Second Method : *The Arithmetical Process.*

$$\begin{array}{r|l}
 & \mathbf{3x^2-\ 4x-2} \\
\cline{2-2}
x+3 & 3x^3+5x^2-14x+4 \\
 & \underline{3x^3+9x^2} \\
 & \qquad -4x^2-14x \\
 & \qquad -4x^2-12x \\
 & \qquad\qquad -\ 2x+4 \\
 & \qquad\qquad \underline{-\ 2x-6} \\
 & \qquad\qquad\qquad \underline{10}
\end{array}$$

Note. The second method is a short-hand form of writing down the argument used in the first method.

Example 28. Divide a^3+b^3 by $a+b$.

$$
\begin{array}{r|l}
 & a^2 \quad - ab \quad + b^2 \\
\hline
a+b & a^3 \qquad\qquad\quad + b^3 \\
 & \underline{a^3 + a^2b} \\
 & \quad - a^2b \\
 & \quad \underline{- a^2b - ab^2} \\
 & \qquad\qquad ab^2 + b^3 \\
 & \qquad\qquad \underline{ab^2 + b^3}
\end{array}
$$

$\therefore$ the quotient is $\mathbf{a^2 - ab + b^2}$, and there is no remainder.

EXERCISE XI. j

Find the products of the following :

1. x^2+3x-1, $x-4$.
2. $3x^2-x+4$, $2x-1$.
3. $x^2+2xy+y^2$, $x-y$.
4. a^2+ab+b^2, $a-b$.
5. x^2+x+1, x^2-x+1.
6. $2y^2-y+3$, $2y^2+y-1$.
7. $3a^2-a-2$, a^2+2a+3.
8. $2b^2+3b-1$, $2b^2-3b-1$.
9. $c^2+2cd+d^2$, $c^2-2cd+d^2$.
10. $y^4-y^2z^2+z^4$, y^2+z^2.
11. $a+b+c$, $a-b+c$.
12. $x-2y+3z$, $x+2y-3z$.

Divide :

13. x^3+x^2-5x-6 by $x+2$.
14. $2x^3-5x^2+x+2$ by $x-1$.
15. $6x^3+x^2-9x-4$ by $2x+1$.
16. $6x^3+5x^2-3x-2$ by $3x-2$.
17. $3y^3+4y^2-17y-9$ by $y+3$.
18. $8z^3+14z^2-23z-6$ by $4z-3$.
19. a^3-1 by $a-1$.
20. b^3+8 by $b+2$.
21. $x^3-3x^2y+3xy^2-y^3$ by $x-y$.
22. $a^4-3a^3b+2b^4$ by $a-b$.
23. $x^4+x^3-8x^2-11x-3$ by x^2-2x-3.
24. $10y^4+11y^3-2y^2-13y-6$ by $2y^2+3y+2$.
25. $6z^4+z^3-16z^2+8$ by $3z^2+2z-4$.
26. $6a^4+15a^3-17a^2-20a+12$ by $2a^2+5a-3$.
27. $2x^4-9x^3-9x^2-13x+6$ by x^2-5x-3.
28. $10y^4-y^3+13y^2-14y+2$ by $5y^2-3y-2$.
29. a^4+1 by $a+1$.
30. b^4+1 by b^2+b+1.
31. x^4-x by x^2+x+1.
32. $8y^4+y$ by $4y^2-2y+1$.

[*For a revision exercise on Ch. XI, see Appendix, Ex. R. 6, p. 268.*]

TEST PAPERS B. 1-10

B. 1

1. If $a = -2$, $b = -5$, find the values of

(i) $(1-a)^2$; (ii) ba^2; (iii) $\dfrac{a+2b}{a-b}$.

2. Simplify (i) $(6x^2 - 4xy) \div (-2x)$; (ii) $(x+a)^2 - (x-a)^2$;

(iii) $\left(x+\dfrac{1}{x}\right)^2 - \left(x-\dfrac{1}{x}\right)^2$.

3. Solve (i) $(2x+1)(2x-1) = (2x+3)(2x-5)$;

(ii) $3y - 4z = 2$, $\frac{1}{3}y + \frac{1}{4}z = 1$.

4. Factorise (i) $x^2 - 7x - 30$; (ii) $ab - ac + cd - bd$.

5. At a school debate, 93 boys vote; the motion is carried by 21 votes. How many opposed the motion ?

B. 2

1. Find (i) the perimeter, (ii) the area of Fig. 175, if all the corners are right-angled, the units being inches.

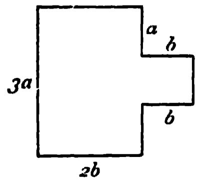


FIG. 175.

If the perimeter is $2p$ inches, express the area in terms of a, p.

2. Simplify (i) $\dfrac{3e-3f}{2f-2e}$; (ii) $\dfrac{a^2-2ab}{a^2b} - \dfrac{ab-b^2}{ab^2}$.

3. Solve (i) $\dfrac{t-5}{7} = \dfrac{7-t}{3}$; (ii) $\frac{1}{2}x + 2y = 5$, $2x - \dfrac{y}{7} + 8\frac{1}{2} = 0$.

4. Factorise (i) $3b^2c - 6bcd + 12bc^2$; (ii) $2y^3 - 18y$.

5. I am thinking of two numbers. If I halve the first, I get one more than the second; if I add 4 to the second, I get one-third of the first. What are the numbers ?

B. 3

1. Simplify (i) $\dfrac{p^2-pq}{pq-q^2}$;

(ii) $(x-1)(x-2) + (x-3)(x-4) - 2(x-6)(x+1)$.

2. The product $p \cdot v$ is constant. If $p = 1\frac{1}{2}$ when $v = 2\frac{1}{2}$, what is the value of v when $p = 3$? What is the value of p when $v^2 = 4$?

3. Solve (i) $\frac{2x+1}{5} - \frac{3x-1}{4} = \frac{1-x}{3}$;
(ii) $y = \frac{1}{3}(x-5)$, $y = \frac{1}{2}(x-3)$.

4. Factorise (i) $x^4 - x^9$; (ii) $2x^2 + 5x - 3$.

5. Find two consecutive odd numbers whose squares differ by 48.

B. 4

1. How far is B from the mid point of AC in Fig. 176 ?

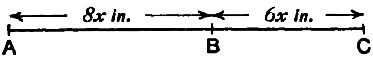


FIG. 176.

If BC is rotated about B through 90°, how far from A is the new position of C ?

2. (i) If $y = \frac{x^2 + 2x}{x+3}$, find the value of x^2y when $x = 2$.

(ii) Simplify $ab\left(\frac{2}{a^2} - \frac{3}{b^2}\right) - 2\left(\frac{a}{b} + \frac{b}{a}\right)$.

3. (i) For what value of c does $x = -1$ satisfy $x^2 - 2cx = c + x$?
(ii) Solve $y = \frac{1}{3}(4x-1)$, $x = \frac{1}{5}(4y-1)$.

4. Factorise (i) $p^2(q+r)^2 - p^4$; (ii) $1 + 3x - 10x^2$.

5. One car takes 7 hours over a journey which another car, travelling 5 miles per hour faster, does in 6 hours. Find the distance.

B. 5

1. If $A = \frac{1}{2}h(a+b)$, express A in terms of h when $2b = 3a = 4\frac{1}{2}h$.

2. (i) Add $\frac{1}{6xyz^2}$ to $\frac{1}{4zyx^2}$; (ii) Simplify $12a^3b^2 \div \left(\frac{2a}{3b}\right)^2$.

3. Solve (i) $\frac{1}{3}(x-4) - \frac{1}{4}(5-x) = \frac{1}{6}(3-4x)$;
(ii) $5x + 11y + 20 = 7x + 4y - 29 = 0$.

4. Factorise (i) $3c^5 - 12cd^2$; (ii) $x^2 + 4x - 77$.

5. The price of eggs having risen by $\frac{1}{2}$d. each, it costs 4d. more to buy 20 eggs than it used to cost to buy 2 dozen eggs. What is the present price ?

B. 6

1. If $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$, express V^2 in terms of S, π.

2. (i) Simplify $\frac{2a-3b}{4} - \frac{a-b}{6} + \frac{2b-a}{3}$.

(ii) Find the value of $(x+y-z)^2 - (x-y-z)^2$ when $x = 2$, $y = -3$, $z = 5$.

3. (i) For what value of x does $\frac{x+3}{5}$ exceed $\frac{7-x}{3}$ by 50 per cent. ?

(ii) Solve $5P = 2Q,\ \frac{1}{4}P - \frac{1}{6}Q + 1 = 0.$

4. Factorise (i) $ax - ay - cx + cy$; (ii) $25x^2 - \frac{1}{4}y^2$.

5. Some stamps are divided equally between 15 boys. If the number of stamps and the number of boys were each increased by 1, each boy would receive 7 stamps less. How many would this be ?

B. 7

1. (i) Simplify $(x+2y)(3y-x) - (2x+y)(y-3x)$.

(ii) What number must be added to $x^2 - 12x$ to make the sum a perfect square ?

2. For what value of c is $x^3 + 2x^2 + cx - 6$ zero, when $x = 2$? For which of the values $x = 1, -1, 3, -3$, is the expression zero, for the same value of c ?

3. (i) Find u in terms of v if $\frac{v+3}{u} = 2\frac{1}{4} + \frac{1}{u}$.

(ii) Solve $x + \frac{1}{2}y = 14,\ \frac{1}{3}x - y = -7.$

4. Factorise (i) $20 + a - a^2$; (ii) $x^2(y-z) - x(y^2 - z^2)$.

5. A motor cyclist averages 30 m.p.h. on ordinary roads, and 10 m.p.h. on roads under repair. His average speed for a run of 50 miles is 20 m.p.h. What length of the road is under repair ?

B. 8

1. What is the sum of the areas of the faces of the rectangular

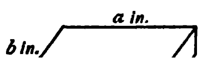


FIG. 177.

block in Fig. 177 ? Express this in terms of c, if $c = \frac{b}{3} = \frac{a}{4}$.

2. (i) Subtract $1 - (p - q)$ from $q - (1 - p)$.

(ii) Simplify $\left(\frac{a}{b} + \frac{b}{a}\right)^2 - \left(\frac{a}{b} - \frac{b}{a}\right)^2$.

3. (i) If $y = 3x - 7$, find x so that the difference between x and y may be 5.

(ii) Solve $\frac{2}{u} + \frac{1}{v} = 3,\ \frac{3}{u} - \frac{2}{v} = 4.$

4. Factorise (i) $6a^2 + 6a - 36$; (ii) $(x+1)^2 - 3(x+1) - 10$.

5. The cost of a book and the postage on it used to be 3 shillings. The price of the book is now 75 per cent. more, and the postal charge has been increased by 50 per cent., so that the present total cost is 5s. 2d. What is the present price of the book ?

B. 9

1. Make a table showing the values of the function, $x^2 - 4x$, for $x = -2, -1, 0, 1, 1{\cdot}5, 2, 2{\cdot}5, 3, 4$.

2. (i) Subtract $na + b$ from $a - kb$ and give the answer in the form $\mathsf{P} \,.\, a + \mathsf{Q} \,.\, b$.

 (ii) Simplify $\dfrac{x^2 - 5x}{x} - \dfrac{3x - x^2}{-x}$.

3. Solve (i) $0{\cdot}25(t+1) - 0{\cdot}4(t-2) = 1{\cdot}02$.

 (ii) $\frac{1}{3}(x+2) - \frac{1}{4}(y-1) = 1,\ 5x = 4y$.

4. Factorise (i) $ax^2 + b^2y - ab^2 - x^2y$;

 (ii) $(5x-1)(x-3) + (x-5)(3-x)$.

5. If the larger of two numbers is divided by the smaller, the quotient and remainder are each 3. If ten times the smaller is divided by the larger, the quotient and remainder are again each 3. What are the numbers ?

B. 10

1. The expression $ax + b$ is equal to 5 when $x = -2$ and is equal to -2 when $x = 5$. For what value of x is the expression equal to x ?

2. (i) Multiply $a - b$ by $b - a$ and add the result to the square of $a + b$.

 (ii) Simplify $\dfrac{p^2 - q^2}{(p - 2q)(2p - q)}$ when $p + 2q = 0$.

3. (i) Solve $x(x+1)(x+7) - x(x+3)^2 = 2(x-3)^2$.

 (ii) Find a, b, c if $a + b = 6,\ b + c = 7,\ c + a = 8$.

4. Factorise (i) $x^2 - 2ax + 3bx - 6ab$;

 (ii) $7a^2 + 9ab - 10b^2$.

5. A train, running between two towns, is 3 minutes late if it averages 45 m.p.h., and is 9 minutes late if it averages 40 m.p.h. Find the distance between the towns, and the proper time for the journey.

[*For additional test papers on Ch. I-XI, see Appendix, Q.* 1-5, *p.* 319.]

CHAPTER XII

QUADRATIC EQUATIONS

Statements as Equations

If all we know about two expressions, A and B, is that $A \,.\, B = 5$, it is impossible to find the value of either, unless we know the value of the other.

But if $$A \,.\, B = 0,$$
we know that **either** $A = 0$ **or** $B = 0$, because the product of two numbers is zero if, and only if, one of them is zero.

Example 1. What can be said about the value of x, if
$$x(2x - 3) = 0\ ?$$

Since the product of x and $2x - 3$ is zero, one of the numbers x and $2x - 3$ must be zero.

$$\therefore\ x = 0 \text{ or } 2x - 3 = 0; \qquad \therefore\ x = 0 \text{ or } 2x = 3;$$
$$\therefore\ x = 0 \text{ or } \frac{3}{2}.$$

Check: If $x = 0$, $x(2x - 3) = 0(-3) = 0$.
If $x = \frac{3}{2}$, $x(2x - 3) = \frac{3}{2}(3 - 3) = \frac{3}{2} \times 0 = 0$.

Do not say $x = 0$ *and* $x = \frac{3}{2}$, because x cannot equal both 0 and $\frac{3}{2}$ at the same time.

Example 2. Combine into a single statement:
$$\text{Either } y = 3 \text{ or } y = -\tfrac{1}{2}.$$

Either $y = 3$ or $2y = -1$; either $y - 3 = 0$ or $2y + 1 = 0$.
$$\therefore\ \text{in either case, } (y - 3)(2y + 1) = 0;$$
$$\therefore\ 2y^2 - 5y - 3 = 0.$$

EXERCISE XII. a

1. If $x = 2$, what is the value of $(x - 2)(x + 10)$?
2. If $y = 5$, what is the value of $(y + 3)(y - 5)$?
3. If $z = 0$, what is the value of $z(2z + 3)$?
4. If $xy = 0$ and $x = 3$, what is y ?
5. If $xy = 0$ and $y = 0$, what can you say about the value of x ?

6. If $xy=1$, can you say anything about the numerical value of x ?

7. What conclusion can be drawn from $(a-b)x=0$?

8. If $(y-3)(z-2)=0$, what do you know about z (i) if $y=2$, (ii) if $y=100$, (iii) if $y=3$?

9. What conclusion can be drawn from the equation $(x-5)(y+3)=0$?

10. If $\dfrac{x+1}{y+2}=0$, can you say anything about the value (i) of x, (ii) of y ?

11. What conclusion can you draw if $(x-5)(x+3)=0$? What do you get if you multiply out ?

12. Combine into a single statement: Either $x=2$ or $x=4$. Multiply out the result.

13. Combine into single statements the following, and multiply out each result.

(i) Either $x=-2$ or $x=-5$; (ii) Either $x=-3$ or $x=4$;
(iii) $y=\pm 7$; (iv) Either $y=6$ or $y=0$;
(v) Either $t=0$ or $t=-8$; (vi) Either $x=2$ or $x=3$ or $x=-1$.

14. What conclusion can you draw if $(3x-2)(2x+5)=0$? What do you get if you multiply out ?

15. Combine into single statements, free from fractions, the following, and multiply out each result :

(i) Either $x=\frac{1}{2}$ or $x=\frac{2}{3}$; (ii) Either $y=-\frac{1}{4}$ or $y=\frac{3}{5}$;
(iii) Either $z=-\frac{3}{4}$ or $z=-\frac{2}{7}$; (iv) Either $x=0$ or $x=-\frac{2}{5}$.

Solve the following equations :

16. $(x-3)(x-7)=0$. 17. $(y+4)(y-5)=0$.

18. $(t+7)(t+2)=0$. 19. $z(z-10)=0$.

20. $(x-3)^2=0$. 21. $(t+5)^2=0$.

22. $(y+1)(y+2)=0$. 23. $n(n+4)=0$.

24. $5x^2=0$. 25. $5(y-2)(y+8)=0$.

26. $(2z-1)(3z-5)=0$. 27. $(3p+1)(p+3)=0$.

28. $7(x-7)(4x-3)=0$. 29. $5(5t+2)(5t-3)=0$.

30. $6x(3x+7)=0$. 31. $(n-2)(n-5)(n+8)=0$.

32. $r(r+3)(r-6)=0$. 33. $y^2(y+4)=0$.

34. $2z(2z+1)(3z-10)=0$. 35. $4x(x-4)^2=0$.

Solution by Factors

Example 3. Solve $x^2 = 25$.

First method. $x^2 - 25 = 0$; $\therefore$ $(x+5)(x-5) = 0$;

$$\therefore \text{ either } x+5=0 \text{ or } x-5=0;$$

$$\therefore x = -5 \textit{ or } 5.$$

Second method. $x^2 = 25$.

Take the square root of each side; the square root of 25 is either $+5$ or -5 because $(+5)(+5)=25$ and $(-5)(-5)=25$;

$$\therefore x = 5 \text{ or } -5, \text{ as before.}$$

The answer is usually written, $x = \pm 5$.

Example 4. Solve $(x+3)(x-5) = 20$.

Multiply out, $\therefore$ $x^2 - 2x - 15 = 20$.

$$\therefore x^2 - 2x - 35 = 0; \quad \therefore (x-7)(x+5) = 0;$$

$$\therefore \text{ either } x-7=0 \text{ or } x+5=0;$$

$$\therefore x = 7 \textit{ or } -5.$$

Check: If $x = 7$, $(x+3)(x-5) = 10 \times 2 = 20$.

If $x = -5$, $(x+3)(x-5) = (-2)(-10) = 20$.

Do not shorten this argument. If you leave out the step, "either $x-7=0$ or $x+5=0$," you may make mistakes in sign.

Example 5. Solve $8x^2 + 6x = 9$.

$$8x^2 + 6x - 9 = 0; \quad \therefore (2x+3)(4x-3) = 0;$$

$$\therefore \text{ either } 2x+3=0 \text{ or } 4x-3=0.$$

$$\therefore 2x = -3 \text{ or } 4x = 3; \quad \therefore x = -\frac{3}{2} \textit{ or } \frac{3}{4}.$$

EXERCISE XII. b

Solve the following equations; in each example, check one answer.

1. $x^2 - 5x + 6 = 0$.	**2.** $x^2 + 5x + 6 = 0$.	**3.** $x^2 - 5x - 6 = 0$.
4. $x^2 + 5x - 6 = 0$.	**5.** $x^2 - 3x + 2 = 0$.	**6.** $x^2 + 3x + 2 = 0$.
7. $x^2 + 9x + 14 = 0$.	**8.** $x^2 - x - 6 = 0$.	**9.** $x^2 + 4x - 5 = 0$.
10. $x^2 - 6x = 0$.	**11.** $x^2 - 9 = 0$.	**12.** $x^2 + 7x = 0$.
13. $2x^2 - 5x + 2 = 0$.	**14.** $3x^2 + x - 2 = 0$.	**15.** $2x^2 - 7x + 6 = 0$.
16. $6x^2 - 7x - 3 = 0$.	**17.** $4x^2 + 13x + 3 = 0$.	**18.** $9x^2 - 30x + 25 = 0$.
19. $x^2 - 3x = 10$.	**20.** $x^2 + 2x = 24$.	**21.** $x^2 + 21 = 10x$.

22. $x^2 = 5x$. 23. $x^2 = 49$. 24. $2x^2 = 6x$.
25. $4x^2 + 1 = 4x$. 26. $2x^2 + x = 15$. 27. $6x^2 = 11x + 7$.
28. $3x^2 = 4x$. 29. $3x^2 + 8x = 3$. 30. $6x^2 + 6 = 13x$.
31. $(x + 1)(x + 2) = 30$. 32. $(x - 3)(x + 2) = 14$.
33. $(2x - 1)(x - 2) = 5$. 34. $(3x + 1)(2x + 3) = 3$.
35. $(x - 3)^2 = 25$. 36. $(x - 1)^2 + (x + 3)^2 = 26$.
37. $(x + 8)(x - 3) = 3x$. 38. $x^2 + \frac{1}{4} = x$.
39. $x + 1 = \dfrac{6}{x}$. 40. $x^3 = 9x$.
41. $(x + 1)(x + 2)(x + 3) = 15(x + 2)$. 42. $x^3 - x = 6(x - 1)$.

[*Note. For additional drill-examples, see Exercise E.P.* 18, *p.* 253.]

The Sum and Product of the Roots

Form the equation whose roots are 3 and 5.

Either $x = 3$ or $x = 5$; either $x - 3 = 0$ or $x - 5 = 0$;

$$\therefore \text{ in either case, } (x - 3)(x - 5) = 0;$$
$$\therefore x^2 - 8x + 15 = 0.$$

Here, the sum of the roots, 5 and 3, equals *the coefficient of x* **with the sign changed**; and the product of the roots equals *the constant term.*

This holds for any quadratic equation, if all the terms are brought to one side and if the coefficient of x^2 is $+1$.

Form the equation whose roots are e, f.

Either $x = e$ or $x = f$; either $x - e = 0$ or $x - f = 0$;

$$\therefore \text{ in either case, } (x - e)(x - f) = 0;$$
$$\therefore x^2 - ex - fx + ef = 0;$$
$$\therefore x^2 - (e + f)x + ef = 0.$$

$\therefore$ *if the coefficient of x^2 is $+1$, and if all terms are brought to one side of the equation,*

(i) **the sum of the roots,** $e + f$, **= the coefficient of x with the sign changed.**

(ii) **the product of the roots,** ef, **= the constant term.**

This gives a useful method for checking answers.

Example 6. Solve $2x^2 = x + 10$.

$$2x^2 - x - 10 = 0; \quad \therefore (2x - 5)(x + 2) = 0;$$
$$\therefore \text{ either } 2x - 5 = 0 \text{ or } x + 2 = 0; \quad \therefore x = 2\tfrac{1}{2} \text{ or } -2.$$

Check: First make the coefficient of x^2 unity and bring all terms to one side.

$$2x^2 - x - 10 = 0 ; \qquad \therefore\ x^2 - \tfrac{1}{2}x - 5 = 0 ;$$

$\therefore$ sum of roots = coefficient of x with sign changed $= +\frac{1}{2}$, and product of roots = constant term $= -5$.

But sum of roots $= 2\frac{1}{2} - 2 = \frac{1}{2}$, and product of roots $= \frac{5}{2} \times (-2) = -5$.

General Statement. If the quadratic equation is

$$\mathbf{ax^2 + bx + c = 0},$$

divide throughout by a, then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Hence **The sum of the roots** $= -\dfrac{\mathbf{b}}{\mathbf{a}}$.

The product of the roots $= \dfrac{\mathbf{c}}{\mathbf{a}}$.

EXERCISE XII. c

What are the roots of the following equations? Then multiply out and verify the relations between the sum and product of the roots and the coefficients.

1. $(x-2)(x-3) = 0$. **2.** $(x+1)(x-5) = 0$. **3.** $(x+2)(x+4) = 0$.

4. $(x-5)(x+5) = 0$. **5.** $(x-3)(x-3) = 0$. **6.** $(x+7)(x-2) = 0$.

7. $(2x-1)(x+3) = 0$. **8.** $(3x+2)(x-2) = 0$. **9.** $(2x+5)(3x+4) = 0$.

Solve the following equations, and check your answers by finding the sum and product of the roots.

10. $x^2 - 8x + 15 = 0$. **11.** $x^2 + 12x + 35 = 0$. **12.** $x^2 - 3x - 28 = 0$.

13. $x^2 + 5x = 24$. **14.** $x^2 - 8x = 9$. **15.** $x^2 + x = 20$.

16. $2x^2 - 3x + 1 = 0$. **17.** $2x^2 + 5x = 12$. **18.** $3x^2 + 5 = 8x$.

19. $4x^2 + 5x = 6$. **20.** $x^2 = 16$. **21.** $x^2 = 11x$.

22. $4x^2 + 4x + 1 = 0$. **23.** $6x^2 = 5x + 6$. **24.** $3x^2 + 2x = 5$.

Write down the sum and the product of the roots of the following equations; check by solving.

25. $x^2 - 9x + 18 = 0$. **26.** $x^2 + 10x + 21 = 0$. **27.** $x^2 + 6x - 16 = 0$.

28. $x^2 + 6x = 27$. **29.** $x^2 - 6x = 40$. **30.** $x^2 - 55 = 6x$.

31. $2x^2 - x = 3$. **32.** $3x^2 = 2x + 8$. **33.** $4x^2 + 3x = 10$.

34. $16x = 4x^2 + 15$. **35.** $5 - x - 6x^2 = 0$. **36.** $x + 2 = 6x^2$.

[*Note. For additional examples, see Appendix, Ex. S.* 12, *p.* 302.]

Graphical Solution

The use of graphs in solving equations has been illustrated in Ch. X; the next two exercises provide further practice.

Example 7. Draw the graph of $2x^2 - 7x - 2$ for values of x from -1 to 4.

When making a table of values, **work by rows.**

x^2	1	0	1	4	9	16	2·25
x	−1	0	1	2	3	4	1·5
$2x^2$ $-7x$	2 7	0 0	2 −7	8 −14	18 −21	32 −28	4·5 −10·5
$2x^2 - 7x - 2$	7	−2	−7	−8	−5	2	−8

First take the values $x = -1, 0, 1, 2, 3, 4$; write above the table the values of x^2 to help you to find $2x^2$.

When these values have been plotted, it becomes clear that the graph can be drawn more accurately if the value, $x = 1·5$, is added to the table; add this at the end.

The required graph is shown in Fig. 178.

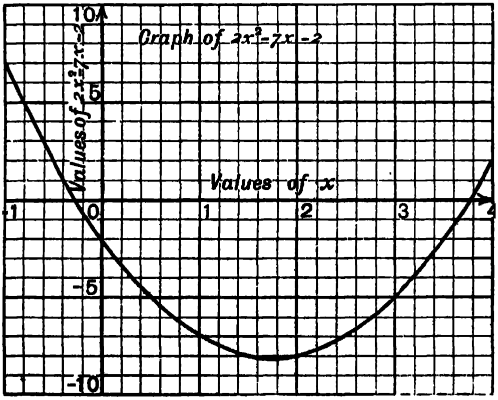


FIG. 178.

The various uses that can be made of this graph are indicated in Ex. XII. d.

EXERCISE XII. d

Use Fig. 178 *to answer the following questions, Nos.* 1-11.

1. What is approximately the value of $2x^2 - 7x - 2$ when $x = 0{\cdot}4$, $1{\cdot}2$, $2{\cdot}5$, $2{\cdot}8$, $3{\cdot}5$, $3{\cdot}9$, $-0{\cdot}5$, $-0{\cdot}9$?

2. What is the least value of $2x^2 - 7x - 2$? For what value of x is $2x^2 - 7x - 2$ least ?

3. State approximately the values of x for which $2x^2 - 7x - 2$ is equal to (i) 2, (ii) 1, (iii) -2, (iv) 0, (v) -5, (vi) -6.

4. For what value of x is $2x^2 - 7x - 2$ equal to 5 ? Is there more than one answer ? If so, why cannot you find it ?

5. Is there a value of x for which $2x^2 - 7x - 2$ equals -10 ?

6. Can you draw a line up the paper about which the curve is symmetrical ? Where will the line cut the x-axis ?

7. For what values of x is $2x^2 - 7x - 2$ negative ?

8. Solve graphically the equation, $2x^2 - 7x - 2 = 0$.
Use the results on p. 199 to say what the sum and the product of the roots should be ; then test the answers obtained from the graph.

9. Solve graphically the following equations, and find in each case the sum of the roots :

(i) $2x^2 - 7x - 2 = 1$; (ii) $2x^2 - 7x - 2 = -2$;
(iii) $2x^2 - 7x - 2 = -6$; (iv) $2x^2 - 7x - 2 = 2$.

10. If $2x^2 - 7x + 5 = 0$, that is, if $2x^2 - 7x = -5$, find the value of $2x^2 - 7x - 2$. Hence solve graphically $2x^2 - 7x + 5 = 0$.

11. Solve, *where possible*, the following equations graphically :

(i) $2x^2 - 7x = -1$; (ii) $2x^2 - 7x = -4$;
(iii) $2x^2 - 7x = 5$; (iv) $2x^2 - 7x = 7$;
(v) $2x^2 - 7x = -6$; (vi) $2x^2 - 7x = -9$;
(vii) $7x - 2x^2 = 3$; (viii) $7x - 2x^2 + 6 = 0$.

EXERCISE XII. e

[Throughout this exercise, use the statement on p. 199 about the sum of the roots of a quadratic equation, as a check.]

1. Draw the graph of $4x - x^2$ from $x = -1$ to $x = 5$, and solve graphically the following equations ?

(i) $4x - x^2 = 2$; (ii) $4x - x^2 = -3$; (iii) $4x - x^2 = 2{\cdot}5$;
(iv) $4x - x^2 = -1$; (v) $x^2 - 4x - 4 = 0$; (vi) $x^2 - 4x + 3 = 0$.

For what values of d has $4x - x^2 = d$ two roots ?

2. Draw the graph of $y=x^2-5x+6$ from $x=0$ to $x=5$.

What are the roots of the following equations ?

(i) $x^2-5x+6=0$; (ii) $x^2-5x+1=0$;

(iii) $x^2-5x+3=0$; (iv) $x^2-5x+4=0$.

Has x^2-5x+6 a greatest value or a least value, and how much is it ?

3. Draw, on the same scale and axes used for No. 2, the graph of $y=x^2-5x+8$ from $x=0$ to $x=5$. Use this graph to solve

(i) $x^2-5x+8=5$; (ii) $x^2-5x+3=0$; (iii) $x^2-5x+2=0$.

Is there a value of x for which $x^2-5x+8=0$? What is the least value of x^2-5x+8 ?

4. Draw the graph of $7x-4x^2$ from $x=-1$ to $x=3$. Solve the following : (i) $7x-4x^2=2$; (ii) $7x-4x^2=-3$;

(iii) $7x-4x^2+10=0$; (iv) $8x^2-14x=11$.

What is the greatest value of $7x-4x^2$?

What can you say about the value of c if there is no value of x which satisfies the equation $7x-4x^2=c$?

Solve, *when possible*, the following equations, Nos. 5-7 :

5. (i) $x^2+4x=1$; (ii) $x^2+4x+2=0$; (iii) $x^2+4x+4=0$; (iv) $x^2+4x+5=0$.

6. (i) $2x^2-3x=3$; (ii) $2x^2-3x+1=0$; (iii) $2x^2-3x+2=0$; (iv) from the same graph, $10x^2-15x+3=0$.

7. (i) $3x^2-5x=2$; (ii) $3x^2-5x+1=0$; (iii) $3x^2-5x+2=0$; (iv) $3x^2-5x+3=0$; (v) $6x^2-10x+1=0$; (vi) $9x^2-15x=4$.

8. Draw the graph of $y=x^2$ from $x=-2$ to $x=+2$.

Draw, with the same scale and axes, the graphs of

(i) $y=x+1$; (ii) $y=\frac{1}{2}x+2$; (iii) $y=\frac{5x}{2}-1$.

What quadratic equations can be solved graphically from this figure ?

If you add to the figure the graph of $y=x-1$, what can you say about the equation, $x^2=x-1$?

9. Draw the graph of $y=x+\frac{12}{x}-6$ from $x=1{\cdot}5$ to $x=8$.

(i) Solve the equations (*a*) $x+\frac{12}{x}-6=1{\cdot}5$; (*b*) $x+\frac{12}{x}=9$.

(ii) What is the least valuo of $x+\frac{12}{x}$, when x is positive ?

(iii) Can you use the graph to solve $x^2+12=8{\cdot}5x$?

10. *Without* making a table of values, state the *sign* of y, where $y=(x+2)(5-x)\equiv 10+3x-x^2$, if x is (i) $+100$; (ii) -100.

Has the graph of $y=10+3x-x^2$ a highest point or a lowest point? Do not draw the graph accurately, but make a rough sketch of it on plain paper.

11. Repeat No. 10 for the function,

$$y=(x+2)(x-5)\equiv x^2-3x-10.$$

Solution by "Completing the Square"

The process of "completing the square" of a quadratic function has been explained on p. 179.

Thus, x^2+6x becomes a perfect square if we add 3^2 to it, $x^2+6x+3^2=(x+3)^2$; x^2-7x becomes a perfect square if we add $(\frac{7}{2})^2$ to it, $x^2-7x+(\frac{7}{2})^2=(x-\frac{7}{2})^2$.

In general, x^2+bx becomes a perfect square if we add $\left(\frac{b}{2}\right)^2$ to it, $x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$.

Example 8. Solve $x^2+6x=27$.

Add 3^2 to each side, $x^2+6x+3^2=27+9$;

$$\therefore\ (x+3)^2=36.$$

Take the square root of each side: the square root of 36 is *either* $+6$ *or* -6.

$$\therefore\ x+3=+6 \text{ or } x+3=-6;$$

$$\therefore\ x=3 \text{ or } -9.$$

For this equation, it is quicker, and therefore better, to use the previous method, thus:

$$x^2+6x-27=0;\quad \therefore\ (x-3)(x+9)=0;$$

$$\therefore\ x-3=0 \text{ or } x+9=0;\quad \therefore\ x=3 \text{ or } -9.$$

There are two distinct stages in the general method of the solution of quadratics by "completing the square." Ex. XII. f is intended to secure familiarity with one of these stages, before the complete process is practised.

EXERCISE XII. f

Solve the following equations, using, where necessary, the approximate square roots given at the end of the Exercise.

1. $(x-5)^2=9$.	**2.** $(x+3)^2=25$.	**3.** $(x-4)^2=1$.
4. $(x+1)^2=\frac{1}{4}$.	**5.** $(x-3)^2=\frac{4}{9}$.	**6.** $(x-2)^2=2\frac{1}{4}$.
7. $(x-2)^2=3$.	**8.** $(x+3)^2=10$.	**9.** $(x+1)^2=2$.

10. $(x-\frac{3}{2})^2=9.$ 11. $(x+\frac{1}{2})^2=4.$ 12. $(x-\frac{5}{2})^2=\frac{9}{4}.$

13. $(x-\frac{3}{4})^2=\frac{25}{16}.$ 14. $(x+\frac{4}{3})^2=\frac{49}{9}.$ 15. $(x+\frac{2}{5})^2=\frac{16}{25}.$

16. $(x-\frac{1}{2})^2=\frac{5}{4}.$ 17. $(x+\frac{1}{3})^2=\frac{10}{9}.$ 18. $(x+\frac{3}{4})^2=\frac{5}{8}.$

[$\sqrt{2} \simeq 1{\cdot}41$, $\sqrt{3} \simeq 1{\cdot}73$, $\sqrt{5} \simeq 2{\cdot}24$, $\sqrt{10} \simeq 3{\cdot}16$.]

The method of "completing the square" should only be used when no simple factors can be found, as in the next two examples.

Example 9. Solve $x^2-7x-5=0$.

$$x^2-7x=5;$$

Add to each side $\left(\frac{7}{2}\right)^2$, $\therefore\ x^2-7x+\left(\frac{7}{2}\right)^2=5+\frac{49}{4};$

$$\therefore\ \left(x-\frac{7}{2}\right)^2=\frac{20+49}{4}=\frac{69}{4};$$

Take the square root of each side,

$$\therefore\ x-\frac{7}{2}=\pm\frac{\sqrt{69}}{2};$$

$$\therefore\ x=\frac{7}{2}+\frac{\sqrt{69}}{2}\ \text{ or }\ x=\frac{7}{2}-\frac{\sqrt{69}}{2}.$$

To find the values of the roots to (say) 2 places of decimals, we find the square root of 69 to 3 places of decimals ; $\sqrt{69}=8{\cdot}307$.

$$\therefore\ x=\frac{7}{2}+\frac{8{\cdot}307}{2}=\frac{7+8{\cdot}307}{2}=\frac{15{\cdot}307}{2}=7{\cdot}65$$

$$\textit{or}\quad x=\frac{7}{2}-\frac{8{\cdot}307}{2}=\frac{7-8{\cdot}307}{2}=-\frac{1{\cdot}307}{2}=-0{\cdot}65.$$

$$\therefore\ x=7{\cdot}65\ \text{ or }\ -0{\cdot}65.$$

The roots, $x=\frac{7}{2}+\frac{\sqrt{69}}{2}$ or $\frac{7}{2}-\frac{\sqrt{69}}{2}$, are usually written more shortly as $\frac{7\pm\sqrt{69}}{2}$.

Check : The sum of the roots $\simeq 7{\cdot}65+(-0{\cdot}65)=7$, which is the coefficient of x with the sign changed.

The product of the roots $\simeq 7{\cdot}65\times(-0{\cdot}65)\simeq -4{\cdot}97$, and this is, approximately, equal to the constant term.

It is impossible to find any whole number or fraction whose square is exactly 69. The square root of 69 is therefore called an **irrational number**; and we call the roots of the equation, $x^2-7x-5=0$, *irrational*. The roots of the equation, $8x^2+6x=9$, (see Example 5, p. 197) are $\frac{3}{4}$, $-\frac{3}{2}$; we call the roots of this equation *rational*, because their *exact* values can be expressed as whole numbers or fractions, i.e. as the *ratio* of two integers.

Example 10. Solve $3x^2+11x+4=0$, giving the roots correct to 2 places of decimals.

$$3x^2+11x=-4\,; \qquad \therefore\ x^2+\frac{11}{3}x=-\frac{4}{3}\,;$$

Add to each side $\left(\frac{11}{6}\right)^2$, $\quad \therefore\ x^2+\frac{11}{3}x+\left(\frac{11}{6}\right)^2=-\frac{4}{3}+\frac{121}{36}\,;$

$$\therefore\ \left(x+\frac{11}{6}\right)^2=\frac{-48+121}{36}=\frac{73}{36}\,;$$

Take the square root of each side,

$$\therefore\ x+\frac{11}{6}=\pm\frac{\sqrt{73}}{6}\,;$$

$$\therefore\ x=-\frac{11}{6}+\frac{\sqrt{73}}{6} \text{ or } -\frac{11}{6}-\frac{\sqrt{73}}{6}.$$

From the tables, $\sqrt{73}=8{\cdot}544$,

$$\therefore\ x=\frac{-11+8{\cdot}544}{6}=-\frac{2{\cdot}456}{6}=-0{\cdot}41$$

or

$$x=\frac{-11-8{\cdot}544}{6}=-\frac{19{\cdot}544}{6}=-3{\cdot}26\,;$$

$$\therefore\ x=-0{\cdot}41 \text{ or } -3{\cdot}26\text{, to 2 places of decimals.}$$

Example 11. Find the sum of the squares of the roots of the equation, $3x^2+11x+4=0$.

Let the roots of the equation be α and β.

From p. 199, $\alpha+\beta=-\frac{11}{3}$ and $\alpha\beta=\frac{4}{3}$;

$$\therefore\ (\alpha+\beta)^2=(-\tfrac{11}{3})^2\,; \quad \therefore\ \alpha^2+2\alpha\beta+\beta^2=\tfrac{121}{9}\,;$$

but $\quad 2\alpha\beta=\frac{8}{3}, \quad \therefore\ \alpha^2+\beta^2=\frac{121}{9}-\frac{8}{3}=\frac{97}{9}\,;$

$$\therefore\ \alpha^2+\beta^2=10\tfrac{7}{9}.$$

Note. This method is simpler than finding each root separately, as in Example 10, and squaring it.

EXERCISE XII. g

What number must be added to the expressions in Nos. 1-9 to make the result a perfect square? Of what is it then the square?

1. x^2-10x.
2. x^2+12x.
3. x^2+3x.
4. x^2-7x.
5. $x^2-\frac{10}{3}x$.
6. $x^2+\frac{4}{7}x$.
7. $x^2+\frac{3}{5}x$.
8. $x^2-\frac{11}{2}x$.
9. $x^2-\frac{7}{9}x$.

The equations in Nos. 10-18 have rational roots. First, solve them by completing the square; then solve by the direct factor method. Compare the length of the work and the results in the two cases.

Check your answers also by calculating the sum and the product of the roots.

10. $x^2-6x=40$. 11. $x^2+5x=14$. 12. $x^2-10x+21=0$.
13. $x^2+13x+22=0$. 14. $x^2+x=12$. 15. $x^2-9x=52$.
16. $3x^2+2x=8$. 17. $2x^2+x=6$. 18. $3x^2-4x=15$.

In solving the following equations, use the direct factor method whenever it is easier to do so. If the roots are not rational, work out each root correct to two places of decimals.

19. $x^2=3$. 20. $2x^2=3$. 21. $2x^2=9$.
22. $x^2+2x=5$. 23. $x^2-2x=5$. 24. $x^2-2x=8$.
25. $x^2+6x=8$. 26. $x^2-6x=16$. 27. $x^2-6x=12$.
28. $x^2-10x+15=0$. 29. $x^2+14x+33=0$. 30. $x^2-12x=25$.
31. $x^2-12x+36=0$. 32. $x^2-20x=33$. 33. $x^2+16x=17$.
34. $x^2+3x=2$. 35. $x^2-5x=3$. 36. $x^2-7x+9=0$.
37. $x^2-7x=30$. 38. $x^2+x=8$. 39. $x^2-x=1$.
40. $x^2+5x+2=0$. 41. $x^2-x=2$. 42. $x^2+15=9x$.
43. $3x^2+4x=2$. 44. $5x^2-8x+2=0$. 45. $3x^2+5x=2$.
46. $3x^2+8x+2=0$. 47. $7x^2-4x=4$. 48. $2x^2+7x=3$.
49. $2x^2-5x=1$. 50. $3x^2+x=2$. 51. $4x^2-6x=3$.
52. $2x^2=3x+6$. 53. $6x^2+4x=3$. 54. $10x^2=2x+5$.

Find the sum of the squares of the roots of the following equations :

55. $x^2-5x=24$. 56. $x^2-5x=3$. 57. $x^2+7x+4=0$.
58. $2x^2+5x=3$. 59. $2x^2-9x=4$. 60. $5x^2-7x+1=0$.

[*Note. For additional drill-examples, see Exercise E.P.* 19, *p.* 254.]

Problems

Example 12. If a stone is projected vertically upwards from the ground with a velocity of 80 feet per second, its height above the ground after t seconds is $(80t-16t^2)$ feet. After what time is it 75 feet above the ground ? Is there a time when the stone is 104 feet above the ground ?

[This formula neglects the effect of air-resistance.]

(i) If it is 75 feet above the ground after t seconds,

$$80t-16t^2=75\;;$$
$$\therefore\ 16t^2-80t+75=0\;;$$
$$\therefore\ (4t-15)(4t-5)=0\;;\quad \therefore\ 4t-15=0 \text{ or } 4t-5=0\;;$$
$$\therefore\ t=\tfrac{15}{4} \text{ or } \tfrac{5}{4}.$$

The stone is therefore 75 feet above the ground, after $1\frac{1}{4}$ sec. and after $3\frac{3}{4}$ sec.

The first answer refers to the upward motion, and the second to the downward motion.

(ii) If it is 104 feet above the ground after t seconds,

$$80t - 16t^2 = 104\ ;\quad \therefore\ 16t^2 - 80t + 104 = 0\ ;$$

$$\therefore\ t^2 - 5t + \tfrac{104}{16} = 0\ ;$$

$$\therefore\ t^2 - 5t + (\tfrac{5}{2})^2 = \tfrac{25}{4} - \tfrac{26}{4} = -\tfrac{1}{4}\ ;$$

$$\therefore\ (t - \tfrac{5}{2})^2 = -\tfrac{1}{4}.$$

But it is impossible to find a number whose square is negative ; $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ and $(-\frac{1}{2}) \times (-\frac{1}{2}) = \frac{1}{4}$.

$\therefore$ there is no value of t such that $(t - \frac{5}{2})^2 = -\frac{1}{4}$, that is, such that $80t - 16t^2 = 104$.

This means that the stone never rises as high as 104 feet above the ground ; and we say that the equation $80t - 16t^2 = 104$ has **no roots.**

Example 13. Is it possible to find a value of x, such that $x^2 - 6x + 13 = 0$?

$$x^2 - 6x = -13.$$

Add to each side 3^2, $\quad \therefore\ x^2 - 6x + 3^2 = -13 + 9\ ;$

$$\therefore\ (x - 3)^2 = -4.$$

But it is impossible to find a number whose square is negative ; $2 \times 2 = 4$ and $(-2) \times (-2) = 4$.

$\therefore$ there is no value of x such that $(x - 3)^2 = -4$, that is, such that $x^2 - 6x + 13 = 0$. We therefore say that the equation $x^2 - 6x + 13 = 0$ has **no roots.**

If the graph of $y = x^2 - 6x + 13$ is drawn, it will be found that the *least* value of y is 4. There is no value of x for which $x^2 - 6x + 13$ is less than 4 ; this also follows from the fact that $x^2 - 6x + 13$ equals $(x - 3)^2 + 4$.

EXERCISE XII. h

1. I think of a number ; then square it and add the original number ; the result is 56. What number did I choose ?

2. The length of a room is 3 ft. more than the breadth, and the floor area is 270 sq. ft. Find the length and breadth.

3. Find two consecutive odd numbers such that the sum of their squares is 650.

4. In Fig. 179, AD . BD is 48 sq. cm., find AD.

- 8 cm. ·

A C B D

FIG. 179.

5. In Fig. 179, $AC^2 + CB^2$ is 40 sq. cm., find AC. Can you find AC so that $AC^2 + CB^2$ is 24 sq. cm. ?

6. The sum of the first n integers 1, 2, 3, 4, 5, ... is $\frac{1}{2}n(n+1)$. How many must be taken to add up to 120 ?

7. A lead sheet, 6 ft. wide, of indefinite length, is bent to form an open rectangular gutter; the area of its cross-section is 640 sq. in. What is the width of the gutter ? (see Fig. 163, p. 163). Can it be bent so that the area of its cross-section is 5 sq. ft. ?

8. A stone is projected vertically upwards so that its height above the ground after t seconds is $(88t - 16t^2)$ feet. After what time is it 105 feet above the ground ?

9. With the data of No. 8, find the length of time before the stone strikes the ground.

10. An n-sided figure has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a figure, if it has 135 diagonals ?

11. A man is x^2 years old and his son is x years old. If the man lives to the age of $13x$, his son will then be x^2 years old. How old is the man now ?

12 The rectangles in Fig. 180 are of equal area, and the units

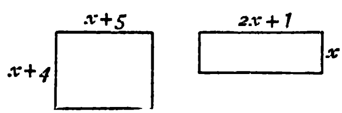


FIG. 180.

of the dimensions are inches. Find the difference between the perimeters.

13. If x^2 degrees E. of N. is the same direction as $3x$ degrees E. of S., find the direction.

14. The units of the dimensions in Fig. 181 are inches and the

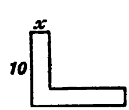


FIG. 181.

area of the figure is 60 sq. in.; find x.

15. If OT is a tangent to the circle in Fig. 182, it can be proved that $OT^2 = OP . OQ$. Find OQ if $PQ = 9$ in., and $OT = 6$ in.

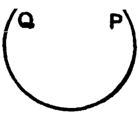


Fig. 182.

16. A marble rolls down a sloping groove and travels $(4t + \frac{1}{2}t^2)$ inches in t seconds. How long will it take to roll 10 feet? What is the meaning of the negative answer?

17. Fig. 183 represents a rectangular lawn with a rectangular flower bed in the middle; the grass is x yards wide on each side and the grass area is 720 sq. yd. Find the dimensions of the flower bed.

18. The sum of two numbers is 9, and the sum of their squares equals twice their product. What are the numbers?

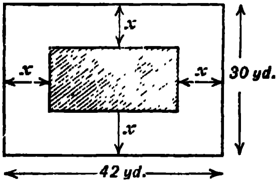


Fig. 183.

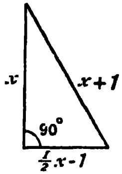


Fig. 184.

19. The product of two consecutive odd numbers exceeds twelve times the intermediate even number by 27. What is the even number?

20. Find the perimeter of the triangle in Fig. 184, the units being inches.

21. I buy a car for £x and sell it for £12. 15s. at a loss of x per cent. Find x.

22. If $\frac{1}{2}(x-1)(x-2)$ and $\frac{1}{3}(x+1)(x+3)$ are consecutive integers, what integers are they?

23. Find the distance from B of a point P on BC in Fig. 185,

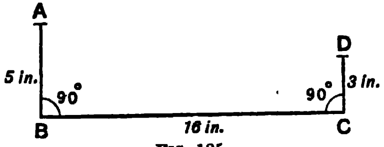


Fig. 185.

such that PA = PD. [The point P is not shown in the figure.]

24. The corners of Fig. 186 are all right-angled, and the measurements are in inches. Find x, if the area of the figure is $\frac{7}{8}$ sq. ft.

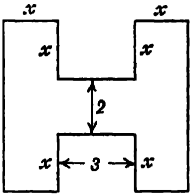


FIG. 186.

25. A cricketer has scored 368 runs. In his next match he is out for 12 and 16 in the two innings and thereby reduces his average by 1. How many times was he out altogether ?

26. My coal bill last year was £28. This year I have to pay 5s. a ton more for my coal. If I can manage to use 2 tons less, my coal bill will remain the same. How much coal did I use last year ?

27. Fig. 187 shows 3 circular arcs touching each other at A, B, C ; the radii of the arcs AC, BC are 3 in., 5 in., and the arc AB is a quadrant. Calculate the radius of arc AB correct to one-tenth of an inch.

28. If the area of Fig. 181 is 40 sq. in., the units in the figure being inches, find the value of x correct to one place of decimals.

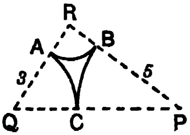


FIG. 187.

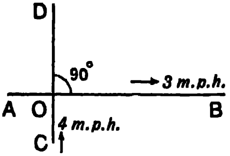


FIG. 188.

29. Fig. 188 shows two cross-roads AOB, COD ; P passes O at 2 p.m. walking at 3 m.p.h. along AB ; Q passes O at 2.30 p.m. walking at 4 m.p.h. along CD. When are P and Q 3 miles apart ?

30. Using the data of No. 8, find whether the stone reaches a height of 125 feet.

After what time is the stone 100 feet above the ground ?

Which of the following equations have no roots ? Do not solve any of them.

31. (i) $x^2-9=0$; (ii) $x^2+4=0$; (iii) $x^2-3=0$.

32. (i) $x^2-2x-3=0$; (ii) $x^2-2x+1=0$; (iii) $x^2-2x+5=0$.

33. (i) $x^2+8x+12=0$; (ii) $x^2+8x+14=0$;
(iii) $x^2+8x+16=0$; (iv) $x^2+8x+17=0$;

34. (i) $x^2-x-1=0$; (ii) $x^2-x+1=0$; (iii) $4x^2-4x+5=0$.

[*Note. For additional problems, see Appendix, Ex. S.* 13, *p.* 304.]

CHAPTER XIII

FRACTIONS

EXERCISE XIII. a (Revision)

Simplify the following expressions. *If there is no simpler form, say so.*

1. $\frac{ax}{ay}$.
2. $\frac{ab}{bc}$.
3. $\frac{p^2}{pq}$.
4. $\frac{-x}{xy}$.
5. $\frac{bc^2}{-bc}$.
6. $\frac{a^2}{b^2}$.
7. $\frac{-y^6}{-y^2}$.
8. $\frac{z^2}{z^3}$.
9. $\frac{abc}{bcd}$.
10. $\frac{4x^2}{2y^2}$.
11. $\frac{-10}{5c}$.
12. $\frac{a^5}{a^3}$.
13. $\frac{-6bc}{-9cd}$.
14. $\frac{2xy^2}{6yz^2}$.
15. $\frac{a^3}{-3c}$.
16. $\frac{b}{b}$.
17. $\frac{9cd^2}{6dc^2}$.
18. $\frac{yz}{z}$.
19. $\frac{3xy}{3yx}$.
20. $\frac{a^3}{3bc}$.
21. $x-\frac{x}{3}$.
22. $\frac{2}{y}-\frac{3}{y}$.
23. $\frac{1}{3z}+\frac{1}{6z}$.
24. $\frac{a}{ab}-\frac{1}{b}$.
25. $1+\frac{x-3}{3}$.
26. $\frac{a}{b}-\frac{a^2}{b^2}$.
27. $\frac{2c}{3}-\frac{c}{6}$.
28. $a-\frac{1}{a}$.
29. $\frac{2a}{3b}-\frac{a}{6b}$.
30. $\frac{x}{3}-\frac{3}{x}$.
31. $\frac{3}{4ab}+\frac{1}{6b^2}$.
32. $\frac{y-4}{8}-\frac{y-3}{6}$.
33. $x\div\frac{x}{y}$.
34. $\frac{a}{b}\div 1$.
35. $\frac{1}{c}\div\frac{-1}{d}$.
36. $1\div\frac{p}{q}$.
37. $-ab\div\frac{-1}{a}$.
38. $\frac{1}{2}xy\times\frac{2y}{-x}$.
39. $\frac{x^4}{y^4}\times\frac{x^2}{y^2}$.
40. $a^3\div\frac{1}{a^2}$.
41. $\frac{ab}{cd}\times\frac{cd^2}{ab^2}$.
42. $\frac{a^2b}{b^2c}\times\frac{bc}{a^2}$.
43. $\frac{6x^2\times 4xy}{9y^2\times 2xz}$.
44. $\frac{5r^6s^6}{15r^2s^2}$.
45. $\frac{a-1}{2b}-\frac{a-2}{3b}-\frac{1}{b}$.
46. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.
47. $\frac{x+y}{6x}-\frac{x-y}{4x}-\frac{2x+y}{12x}$.
48. $1+\frac{2b-c}{4c}-\frac{b+3c}{10c}$.
49. $\frac{a^2-2a}{\quad}-\frac{a^2-a^3}{\quad}-a$.
50. $\frac{x^2-xy}{xy}-\frac{yz-z^2}{yz}$.
51. $y\left(1-\frac{1}{y}\right)-z\left(1-\frac{1}{z}\right)$.
52. $\frac{1}{r}(2r^2-rs)-\frac{1}{s}(2s^2+rs)$.

Reduction to Lowest Terms

Fractions in Algebra are simplified by precisely the same methods as are used in Arithmetic.

Example 1. Express $\frac{x^2+xy}{x^2-y^2}$ in its lowest terms.

$$\frac{x^2+xy}{x^2-y^2}=\frac{x(x+y)}{(x+y)(x-y)};$$

Divide numerator and denominator by $(x+y)$,

$$\therefore\ \frac{x^2+xy}{x^2-y^2}=\frac{x}{x-y}.$$

This fraction cannot be *reduced* any further, because there is no factor common to x and $x-y$.

If we divide numerator and denominator by x, we shall have

$\frac{x}{x-y}=\frac{1}{1-\frac{y}{x}}$; but *this is a less simple form.*

Example 2. Simplify (i) $\frac{x+\frac{1}{2}}{x+\frac{1}{3}}$; (ii) $\frac{1}{\frac{1}{u}+\frac{1}{v}}$.

(*i*) Multiply numerator and denominator by 6.

$$\frac{x+\frac{1}{2}}{x+\frac{1}{3}}=\frac{6x+3}{6x+2}=\frac{3(2x+1)}{2(3x+1)}.$$

(ii) Multiply numerator and denominator by uv.

$$\frac{1}{\frac{1}{u}+\frac{1}{v}}=\frac{uv}{\frac{uv}{u}+\frac{uv}{v}}=\frac{uv}{v+u}.$$

Or

$$\frac{1}{\frac{1}{u}+\frac{1}{v}}=1\div\left(\frac{1}{u}+\frac{1}{v}\right)=1\div\left(\frac{v+u}{uv}\right)$$

$$=1\times\frac{uv}{v+u}=\frac{uv}{v+u}.$$

Example 3. Simplify $\frac{b^2-a^2}{2a^2+ab-3b^2}$.

$$\frac{b^2-a^2}{2a^2+ab-3b^2}=\frac{(b+a)(b-a)}{(a-b)(2a+3b)};$$

but $\qquad b+a=a+b$ and $b-a=-(a-b)$;

$$\therefore\ \text{the fraction}=\frac{-(a+b)(a-b)}{(a-b)(2a+3b)}.$$

Divide numerator and denominator by $(a-b)$,

$$\therefore \text{ the fraction} = -\frac{a+b}{2a+3b}.$$

This fraction cannot be simplified further, because there is no factor common to $a+b$ and $2a+3b$.

EXERCISE XIII. b

Express the following fractions in their simplest form. *If there is no simpler form, say so.*

1. (i) $\dfrac{17-8}{17+8}$; (ii) $\dfrac{30+40}{1200}$; (iii) $\dfrac{5^2+4^2}{5^2-4^2}$.

2. (i) $\dfrac{6^2+6}{6^2+1}$; (ii) $\dfrac{5^2+18}{50-18}$; (iii) $\dfrac{8^2}{8^2-24}$.

3. (i) $\dfrac{a^2+ab}{ab}$; (ii) $\dfrac{a^2+ab}{a^2-ab}$; (iii) $\dfrac{b^2}{b^2-bc}$.

4. $\dfrac{4p+4q}{6p-6q}$. 5. $\dfrac{8u+8v}{6u+6v}$. 6. $\dfrac{r+r^2}{s+rs}$.

7. $\dfrac{4b+c}{4b-c}$. 8. $\dfrac{4m+4n}{4m-4n}$. 9. $\dfrac{xy-xz}{xy+xz}$.

10. $\dfrac{r^2-s^2}{(r-s)^2}$. 11. $\dfrac{p^2-pq}{pq-q^2}$. 12. $\dfrac{a^2+b^2}{a^2-b^2}$.

13. $\dfrac{x}{x^2+xy}$. 14. $\dfrac{a^2-1}{a^2-a}$. 15. $\dfrac{b-c}{c-b}$.

16. $\dfrac{n^2-9}{n+3}$. 17. $\dfrac{r^2-s^2}{s-r}$. 18. $\dfrac{2x+3y}{3x+2y}$.

19. $\dfrac{(a-b)^2}{b-a}$. 20. $\dfrac{c+d}{(d+c)^2}$. 21. $\dfrac{a-b}{4b-4a}$.

22. $\dfrac{r^2-r-6}{r-3}$. 23. $\dfrac{x-2}{x^2+x-6}$. 24. $\dfrac{y^2-z^2}{(z-y)^2}$.

25. $\dfrac{3x+2(x+y)}{x(x+y)}$. 26. $\dfrac{(2a-2b)^2}{(3a-3b)^2}$. 27. $\dfrac{x^2+4x-32}{xy-4y}$.

28. $\dfrac{xy+ax}{xy+ay}$. 29. $\dfrac{x^2-6x+5}{1-x}$. 30. $\dfrac{3a^3b-3ab^3}{a^2+ab}$.

31. $\dfrac{2(a+b)+3(a-b)}{(a+b)(a-b)}$. 32. $\dfrac{t^2+t}{t^2+1}$. 33. $\dfrac{2x^2-7x+3}{(2x-6)^2}$.

34. $\dfrac{3x^2+5xy-2y^2}{4x^2+7xy-2y^2}$. 35. $\dfrac{(1-a)(1-b)}{(a-1)(b-1)}$. 36. $\dfrac{6p^2+5pq-6q^2}{6p^2-pq-2q^2}$.

37. $\dfrac{1}{a+\frac{1}{2}}$. 38. $\dfrac{\frac{1}{2}a}{b-\frac{1}{2}}$. 39. $c+\dfrac{1}{d}$

40. $\dfrac{\frac{x}{2}-y}{x^2-4y^2}$.
41. $\dfrac{t-\frac{1}{t}}{(t-1)^2}$.
42. $\dfrac{1-\frac{1}{c^2}}{c+\frac{1}{c}+2}$.

43. $\dfrac{\frac{1}{a}+\frac{1}{b}}{a+b}$.
44. $\dfrac{p+\frac{1}{q}}{q+\frac{1}{p}}$.
45. $\dfrac{x+1-\frac{2}{x}}{x^2-x-6}$.

Addition and Subtraction

Two fractions are called *equivalent*, if either can be reduced to the other by dividing (or multiplying) numerator and denominator by equal expressions.

Thus, $\dfrac{x}{x-y}$ is equivalent to $\dfrac{x(x+y)}{(x-y)(x+y)}$ or $\dfrac{x^2+xy}{x^2-y^2}$.

As in Arithmetic, fractions are added and subtracted by replacing any of them by equivalent fractions, so arranged that all the denominators are equal.

Example 4. Simplify $\dfrac{2}{x}+\dfrac{1}{x+y}$.

Make the denominator of each fraction $x(x+y)$.

$$\frac{2}{x}+\frac{1}{x+y}=\frac{2(x+y)}{x(x+y)}+\frac{x}{x(x+y)}=\frac{2(x+y)+x}{x(x+y)}$$
$$=\frac{2x+2y+x}{x(x+y)}=\frac{3\mathbf{x}+2\mathbf{y}}{\mathbf{x}(\mathbf{x}+\mathbf{y})}.$$

Leave the denominator in factors.

Example 5. Simplify $\dfrac{a+b}{a^2-ab}-\dfrac{a+2b}{a^2-b^2}$.

The expression $=\dfrac{a+b}{a(a-b)}-\dfrac{a+2b}{(a+b)(a-b)}$.

The L.C.M. of $a(a-b)$ and $(a+b)(a-b)$ is $a(a-b)(a+b)$; make the denominator of each fraction $a(a-b)(a+b)$.

$$\text{The expression}=\frac{(a+b)(a+b)}{a(a-b)(a+b)}-\frac{(a+2b)a}{a(a+b)(a-b)}$$
$$=\frac{(a^2+2ab+b^2)-(a^2+2ab)}{a(a+b)(a-b)}$$
$$=\frac{a^2+2ab+b^2-a^2-2ab}{a(a+b)(a-b)}=\frac{\mathbf{b}^2}{\mathbf{a}(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})}.$$

Example 6. Simplify $x - \dfrac{x^2}{x-y}$.

$$x = \frac{x}{1} = \frac{x(x-y)}{x-y};$$

$$\therefore \text{ the expression} = \frac{x(x-y)}{x-y} - \frac{x^2}{x-y} = \frac{x(x-y)-x^2}{x-y}$$

$$x - y \qquad x - y$$

$$= -\frac{\mathbf{xy}}{\mathbf{x-y}}.$$

We may write this answer, $\dfrac{xy}{y-x}$

because
$$\frac{-xy}{x-y} = \frac{+xy}{(-1)(x-y)} = \frac{xy}{y-x}.$$

EXERCISE XIII. c

Copy and complete the following :

1. (i) $\dfrac{3}{5} = \dfrac{?}{25}$; (ii) $\dfrac{3x}{y} = \dfrac{?}{xy} = \dfrac{?}{y^2}$; (iii) $\dfrac{3x}{x+y} = \dfrac{?}{y(x+y)}$.
2. (i) $\dfrac{1}{a} = \dfrac{?}{a^2} = \dfrac{?}{ab}$; (ii) $\dfrac{1}{a-b} = \dfrac{?}{a(a-b)} = \dfrac{?}{a^2-b^2} = \dfrac{?}{(a-b)^2}$.
3. (i) $\dfrac{b}{c} = \dfrac{?}{2c} = \dfrac{?}{c+c^2}$; (ii) $\dfrac{pq}{p+2q} = \dfrac{?}{q(p+2q)} = \dfrac{?}{(p+2q)(p+3q)}$.
4. (i) $t = \dfrac{?}{t+2} = \dfrac{?}{s-t}$; (ii) $\dfrac{x+y}{x(x-y)} = \dfrac{?}{xy(x-y)^2} = \dfrac{?}{x(x-y)(x-2y)}$.

Find the L.C.M. of the following :

5. $2x - 2y$, $3x - 3y$.
6. $4a + 4b$, $6a - 6b$.
7. $3x^2 - 3$, $5x + 5$.
8. $p^2 - pq$, $pq - q^2$.
9. $4y - 4z$, $10z - 10y$.
10. $x^3y + x^2y^2$, $x^2y^2 - xy^3$.
11. $a^2 - b^2$, $a^2 - 2ab + b^2$.
12. $x^2 + x - 6$, $x^2 - 7x + 10$.

Replace the following pairs of fractions by equivalent fractions with denominators equal to each other.

13. $\dfrac{1}{x+1}$; $\dfrac{1}{x-1}$.
14. $\dfrac{1}{a}$; $\dfrac{1}{a+b}$.
15. $\dfrac{b}{b+c}$; $\dfrac{c}{b-c}$.
16. $\dfrac{x^2}{x^2-y^2}$; $\dfrac{x}{x-y}$.
17. $\dfrac{3}{2a+2}$; $\dfrac{5}{3a+3}$.
18. $\dfrac{5}{6z}$; $\dfrac{z+2}{2z+6}$.

Express the following as single fractions in their lowest terms. *Leave the denominators in factors.*

19. $\dfrac{1}{x+2} + \dfrac{1}{x-2}$.
20. $\dfrac{1}{x+3} - \dfrac{1}{x-3}$.
21. $\dfrac{3}{x+1} - \dfrac{2}{x-1}$.

22. $\dfrac{a}{a-b}-\dfrac{a}{a+b}$.

23. $\dfrac{1}{b}+\dfrac{1}{b-c}$.

24. $r-\dfrac{rs}{r+s}$.

25. $\dfrac{p}{p+q}+\dfrac{q}{p-q}$.

26. $1-\dfrac{x}{x-y}$.

27. $\dfrac{1}{2}+\dfrac{[illegible]}{x-2y}$.

28. $a+\dfrac{ab}{a-b}$.

29. $\dfrac{1}{p}-\dfrac{q}{p^2+pq}$.

30. $\dfrac{1}{r}-\dfrac{r}{r^2-s^2}$.

31. $\dfrac{1}{t^2}-\dfrac{1}{t^2+t}$.

32. $\dfrac{b}{2b+c}-\dfrac{1}{2}$.

33. $\dfrac{1}{x}-\dfrac{2y}{xy-y^2}$.

34. $\dfrac{1}{2a-2b}-\dfrac{1}{3a-3b}$.

35. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}$.

36. $\dfrac{y}{x^2-xy}-\dfrac{x}{xy-y^2}$.

37. $\dfrac{1}{y}+\dfrac{1}{z}-\dfrac{1}{y+z}$.

38. $y+z+\dfrac{z^2}{y-z}$.

39. $\dfrac{c}{b^2+bc}+\dfrac{b-c}{(b+c)^2}$.

40. $\dfrac{1}{x-1}+\dfrac{2}{x^2+2x-3}$.

41. $\dfrac{1}{x-2}-\dfrac{7}{2x^2-x-6}$.

42. $\dfrac{3}{a+b}-\dfrac{2(a-2b)}{a^2-b^2}$.

43. $\dfrac{1}{x^2+x-6}-\dfrac{1}{x^2-x-12}$.

44. $\dfrac{t-3}{t^2-3t-4}-\dfrac{t-1}{t^2-t-2}$.

45. $\dfrac{p+q}{p^2-4q^2}-\dfrac{p+2q}{3p^2-5pq-2q^2}$.

Multiplication and Division

Example 7. Simplify $\dfrac{x^2-xy}{x+y}\times\dfrac{x^2-4y^2}{2x^2-5xy+2y^2}\div\dfrac{y^2-xy}{2x-y}$.

$$\text{The expression}=\frac{x(x-y)}{x+y}\times\frac{(x+2y)(x-2y)}{(2x-y)(x-2y)}\times\frac{2x-y}{y(y-x)}$$
$$=\frac{x(x-y)}{x+y}\times\frac{(x+2y)(x-2y)}{(2x-y)(x-2y)}\times\frac{2x-y}{-y(x-y)}.$$

Divide numerator and denominator by the common factors, $(x-y)$, $(x-2y)$, $(2x-y)$.

$$\therefore\ \text{the expression}=\frac{x(x+2y)}{-y(x+y)}=-\frac{\mathbf{x(x+2y)}}{\mathbf{y(x+y)}}.$$

EXERCISE XIII. d

Simplify the following:

1. $\dfrac{2a}{2a+b}\times\dfrac{2ab+b^2}{6a^2}$.

2. $\dfrac{x^2-x}{2x+2}\times\dfrac{6x}{x^2-1}$.

3. $\dfrac{x^2-5x+6}{x}\times\dfrac{x+2}{x^2-3x}$.

4. $\dfrac{b-c}{c+b}\div\dfrac{c-b}{b+c}$.

5. $\frac{1}{2}x \times \frac{x^2-1}{x^2-x}$.

6. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{a}$.

7. $\frac{a-b}{a-c} \times \frac{b-c}{b-a} \times \frac{c-a}{c-b}$.

8. $\frac{x^2-3x}{2x^2+7x+3} \div \frac{x^2-5x+6}{2x^2-3x-2}$.

9. $\frac{a^2+2ab}{ab} \div (a^2-4b^2)$.

10. $\frac{x^2-y^2}{x^2-2xy+y^2} \times \frac{1}{xy+y^2}$.

11. $\frac{x^2+x-6}{x+1} \times \frac{2x^2+x-1}{x+3}$.

12. $\frac{a^2-4a+4}{6a-2} \times \frac{9a^2-6a+1}{3a^2-12}$.

13. $\frac{3x-x^2-2}{x^2-2x-3} \times \frac{6x-x^2-9}{x^2-5x+6}$.

14. $\frac{x^2+(a+1)x+a}{x^2+2ax+a^2} \times \frac{a+x}{a+1}$.

15. $\frac{x^2+2x-8}{5-x} \times \frac{x^2-8x+15}{1-x} \times \frac{x^2+2x-3}{2-x}$.

16. $\frac{b^2+3bc+2c^2}{b^3-3b^2c} \times \frac{b^3+b^2c}{b^2+5bc+6c^2} \div \frac{b^2+3bc}{b-3c}$.

17. $\left(x+\frac{12}{x}-7\right) \times \left(x+\frac{5}{x}-6\right) \div \left(x+\frac{3}{x}-4\right)$.

18. $\frac{a^2-4ab-5b^2}{a^2-2ab-15b^2} \times \frac{a^2}{a^2-4ab} \div \frac{a^2-ab-2b^2}{a^2-ab-12b^2}$.

[*Note. For additional drill-examples, see Exercise E.P.* 20, *p.* 254.]

Further Simplification

Example 8. Find the L.C.M. of

$$x^2+x-6,\ x^2-x-12,\ 16-x^2,\ 2x-x^2.$$

Use the method of factors, as in Arithmetic.

$$x^2+x-6=(x+3)(x-2)\ ;\ x^2-x-12=(x+3)(x-4)\ ;$$
$$16-x^2=(4+x)(4-x)=-(x+4)(x-4)\ ;$$
$$2x-x^2=x(2-x)=-x(x-2).$$
$$\therefore \text{ the L.C.M. is } \mathbf{x(x-2)(x+3)(x-4)(x+4)}.$$

Example 9. Simplify $\left(\frac{a}{b}-\frac{b}{a}\right)\left(1+\frac{b}{a-b}\right) \div \left(1-\frac{b}{a+b}\right)$.

First reduce the contents of each bracket to a single fraction.

$$\text{The expression} = \left(\frac{a^2-b^2}{ab}\right) \cdot \left(\frac{a-b+b}{a-b}\right) \div \left(\frac{a+b-b}{a+b}\right)$$
$$= \frac{(a+b)(a-b)}{ab} \cdot \frac{a}{a-b} \times \frac{a+b}{a}$$
$$= \frac{(\mathbf{a}+\mathbf{b})^2}{\mathbf{ab}}.$$

Example 10. Simplify $\dfrac{3(x+y)}{x^2+xy-2y^2}+\dfrac{3x+y}{y^2-x^2}$.

The expression

$$=\frac{3(x+y)}{(x-y)(x+2y)}+\frac{3x+y}{(y+x)(y-x)}$$

$$=\frac{3(x+y)}{(x-y)(x+2y)}-\frac{3x+y}{(x+y)(x-y)}, \text{ since } (y-x)=-(x-y),$$

$$=\frac{3(x+y)(x+y)}{(x+y)(x-y)(x+2y)}-\frac{(3x+y)(x+2y)}{(x+y)(x-y)(x+2y)}$$

$$=\frac{3(x^2+2xy+y^2)-(3x^2+7xy+2y^2)}{(x+y)(x-y)(x+2y)}$$

$$=\frac{3x^2+6xy+3y^2-3x^2-7xy-2y^2}{(x+y)(x-y)(x+2y)}$$

$$=\frac{y^2-xy}{(x+y)(x-y)(x+2y)}=\frac{-y(x-y)}{(x+y)(x-y)(x+2y)}$$

$$=-\frac{\mathbf{y}}{(\mathbf{x}+\mathbf{y})(\mathbf{x}+2\mathbf{y})},$$

Fractions should always be expressed in their lowest terms. Before showing up an answer, see whether the numerator and denominator have any common factor; if so, simplify the fraction.

EXERCISE XIII. e

Copy and complete the following :

1. $\dfrac{a}{b-c}=\dfrac{?}{c-b}=\dfrac{?}{c^2-b^2}$.

2. $\dfrac{x-y}{x+y}=\dfrac{(y-x)^2}{?}=\dfrac{?}{y^2-x^2}$.

3. $\dfrac{1}{(b-c)(a-c)}=\dfrac{?}{(a-b)(b-c)(c-a)}$.

4. $\dfrac{b-c}{(a-b)(a-c)}=\dfrac{?}{(a-b)(b-c)(c-a)}$.

Find the L.C.M. of the following :

5. x^2+x-2, $x^2-3x-10$, x^2-4x-5.
6. a^2+a, a^2+2a+1, $a^2+ab+a+b$.
7. x^2-xy, y^2-xy, $2xy-x^2-y^2$.
8. x^3+x^2, x^2+2x-8, $2+x-x^2$, $x-x^3$.

Simplify the following :

9. $\dfrac{1}{c-d}+\dfrac{1}{d-c}$.

10. $\dfrac{p}{p-q}+\dfrac{q}{q-p}$.

11. $\dfrac{x+2y}{x^2-y^2}+\dfrac{2y}{xy-x^2}$.

12. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(c-a)(c-b)}$.

3. $\dfrac{5}{a+b}-\dfrac{4}{a-b}-\dfrac{8a}{b^2-a^2}.$

14. $\dfrac{r+2s}{r^2-rs}+\dfrac{s+2r}{s^2-sr}.$

15. $\dfrac{z+1}{z^2-4z+3}-\dfrac{z-3}{z^2-1}.$

16. $\dfrac{3}{y+3}+\dfrac{4}{y+4}-\dfrac{7}{y+7}.$

17. $\dfrac{a}{b+\frac{1}{2}}+\dfrac{a}{b-\frac{1}{2}}-\dfrac{2a}{b}.$

18. $\dfrac{1}{a+\dfrac{1}{b}}+\dfrac{1}{b+\dfrac{1}{a}}-\dfrac{1}{\frac{1}{2}a+\frac{1}{2}b}.$

19. $\dfrac{1}{[illegible]}+\dfrac{1}{[illegible]}+\dfrac{x}{[illegible]}-\dfrac{x}{[illegible]}-1.$

20. $\left(1+\dfrac{b}{a}\right)\div\left(1+\dfrac{a}{b}\right).$

21. $\left(1-\dfrac{c}{d}\right)\div\left(1-\dfrac{d}{c}\right).$

22. $\left(x+\dfrac{x}{y}\right)\div\left(x-\dfrac{x}{y}\right).$

23. $\left(\dfrac{1}{p}-\dfrac{1}{q}\right)\div\left(\dfrac{p}{q}-\dfrac{q}{p}\right).$

24. $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)(x+y)\div\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right).$

25. $\left(a-\dfrac{10}{a}-3\right)\div(\tfrac{1}{2}a+1).$

26. $\dfrac{4}{x-2}-\dfrac{1}{x+3}-\dfrac{5}{x^2+x-6}.$

27. $\dfrac{b+c}{b-c}+\dfrac{b-c}{b+c}+\dfrac{4bc}{c^2-b^2}.$

28. $\left(\dfrac{x}{x-y}+\dfrac{y}{y-x}\right)\left(\dfrac{x-y}{x}-\dfrac{y-x}{y}\right).$

29. $\left(\dfrac{a}{b}+\dfrac{a+b}{a-b}\right)\div\left(\dfrac{b}{a}+\dfrac{a-b}{a+b}\right).$

30. $\dfrac{a}{2a+b}+\dfrac{2b}{a-2b}-\dfrac{5ab}{2a^2-3ab-2b^2}.$

31. $\left\{\dfrac{p}{(p-1)^2}+\dfrac{1}{1-p}\right\}\times\left(\dfrac{1}{p}-1\right).$

32. $\dfrac{3}{x^2+x-2}-\dfrac{2}{x^2+2x-3}+\dfrac{1}{x^2+5x+6}.$

33. $\dfrac{2}{3x^2-4xy+y^2}+\dfrac{2}{2x+y}-\dfrac{5}{y^2-xy-6x^2}.$

34. $\left(\dfrac{x}{y}+\dfrac{y}{x}-2\right)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\div\left(\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}\right).$

35. $\left(\dfrac{1}{1-a}-\dfrac{2a}{1-a^2}\right)\times\dfrac{a^2-2a-3}{a^2-6a+9}.$

36. $\dfrac{b+c}{(b-a)(c-a)}+\dfrac{c+a}{(c-b)(a-b)}+\dfrac{a+b}{(a-c)(b-c)}.$

Equations

Example 11. Solve $\dfrac{x}{x-2}-\dfrac{3}{x+1}=1.$

Multiply each side by $(x-2)(x+1)$.

$$\therefore\ x(x+1)-3(x-2)=(x-2)(x+1).$$
$$\therefore\ x^2+x-3x+6=x^2-x-2\,;$$
$$\therefore\ -x=-8\,;\quad \therefore\ x=8.$$

Check. If $x=8$, left side $=\frac{8}{6}-\frac{3}{9}=\frac{4}{3}-\frac{1}{3}=1.$

Example 12. Solve $\dfrac{5}{x-6}-\dfrac{4}{x-4}=\dfrac{3}{x+2}.$

Multiply each side by $(x-6)(x-4)(x+2)$.

$$\therefore\ 5(x-4)(x+2)-4(x-6)(x+2)=3(x-6)(x-4).$$
$$\therefore\ 5(x^2-2x-8)-4(x^2-4x-12)=3(x^2-10x+24).$$
$$\therefore\ 5x^2-10x-40-4x^2+16x+48=3x^2-30x+72.$$
$$\therefore\ -2x^2+36x-64=0\,;\quad \therefore\ x^2-18x+32=0\,;$$
$$\therefore\ (x-2)(x-16)=0\,;\quad \therefore\ x-2=0 \text{ or } x-16=0\,;$$
$$\therefore\ x=2 \text{ or } 16.$$

Check. If $x=2$, left side $=\dfrac{5}{-4}-\dfrac{4}{-2}=-1\frac{1}{4}+2=\frac{3}{4}$;
right side $=\frac{3}{4}$.
If $x=16$, left side $=\frac{5}{10}-\frac{4}{12}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$;
right side $=\frac{3}{18}=\frac{1}{6}$.

EXERCISE XIII. f

Solve the following equations :

1. $\dfrac{1}{x}=2\frac{1}{2}.$
2. $\dfrac{1}{y-1}=\dfrac{2}{7}.$
3. $\dfrac{1}{t}-\dfrac{5}{3t}=\dfrac{5}{12}.$
4. $\dfrac{2}{p-1}=\dfrac{5}{p}.$
5. $\dfrac{1}{z}+\dfrac{1}{z+2}=0.$
6. $\dfrac{x-3}{x+6}=4.$
7. $r=\dfrac{1}{r}.$
8. $r=\dfrac{6}{r+1}.$
9. $r-1=\dfrac{2}{r+1}.$
10. $\dfrac{1}{t}-\dfrac{2}{t+1}+\dfrac{1}{t+3}=0.$
11. $\dfrac{1}{y-2}+\dfrac{3}{y-3}+\dfrac{5}{2-y}=0.$
12. $\dfrac{2x-1}{3x+1}=\dfrac{6x-1}{9x-3}.$
13. $\dfrac{3}{p}=\dfrac{2}{p+1}+\dfrac{2}{3p-1}.$
14. $3x+5=\dfrac{3x^2+5}{5x+3}.$
15. $\dfrac{3}{x+4}-\dfrac{2}{x+3}=\dfrac{1}{x+1}.$
16. $\dfrac{5}{x-8}=\dfrac{2}{x-3}+\dfrac{3}{x+2}.$
17. $\dfrac{6}{x+1}+\dfrac{1}{1-x}=\dfrac{2}{x}.$
18. $\dfrac{1}{2y-3}=\dfrac{5}{y}+\dfrac{3}{2y^2-3y}.$
19. $\dfrac{1}{x-2}-\dfrac{1}{x+1}=\dfrac{1}{6}.$

20. $\dfrac{t-1}{t+2}+\dfrac{t+3}{t-2}=2.$

21. $\dfrac{r-1}{r+2}+\dfrac{r+3}{r^2-4}=\dfrac{3r-1}{5(r-2)}.$

22. $\dfrac{3p-1}{p-1}-\dfrac{2p+1}{p+1}=1.$

23. $\dfrac{3x^2}{x^2-1}=\dfrac{4}{x-1}.$

24. $\dfrac{4}{x-3}+\dfrac{3x-3}{x^2-x-6}=\dfrac{10x+10}{3x+6}.$

25. $y-1=\dfrac{y^2+2}{y-1}+\dfrac{y+2}{y-6}.$

26. $\dfrac{1}{x+1}+\dfrac{1}{x+2}+\dfrac{1}{x+3}=0$, correct to one place of decimals.

27. $\dfrac{2x+2}{x-1}-\dfrac{x-1}{x+1}=\dfrac{x}{x-2}$, correct to one place of decimals.

28. $\dfrac{1}{x-2}+\dfrac{2}{x-1}=\dfrac{2}{x-3}$, correct to two places of decimals.

Miscellaneous Fractions and Equations

EXERCISE XIII. g

Simplify the given expressions, and solve the given equations:

1. $\dfrac{1}{2}-\dfrac{1}{x-2}.$

2. $\dfrac{1}{2}-\dfrac{1}{x-2}=\dfrac{1}{4}.$

3. $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}.$

4. $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{1}{9}.$

5. $\dfrac{x}{3}-\dfrac{x+1}{5}.$

6. $\dfrac{x}{3}-\dfrac{x+1}{5}=1.$

7. $1-\dfrac{1}{x+1}.$

8. $2+\dfrac{1}{2x+1}=1.$

9. $x+3-\dfrac{x^2}{x-3}.$

10. $\dfrac{1}{3x}-\dfrac{1}{3x+3}.$

11. $\dfrac{1}{3x}-\dfrac{1}{3x+3}=\dfrac{1}{4x}.$

12. $\dfrac{x}{x+2}-\dfrac{1}{x}=1.$

13. $\dfrac{x}{x-1}-\dfrac{1}{x}-1.$

14. $\dfrac{x}{2x-4}-\dfrac{x}{3x-6}.$

15. $\dfrac{x}{x-1}+\dfrac{1}{x}=1.$

16. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)\div\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right).$

17. $\left(\dfrac{a}{b}-1\right)\div\left(\dfrac{b}{a}-1\right).$

18. $\left(1-\dfrac{1}{x-1}\right)\left(1+\dfrac{1}{x-2}\right).$

19. $x+4-\dfrac{x^2}{x-4}=1.$

20. $\dfrac{a^2-2ab+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+2ab+b^2}.$

21. $\left(c-2-\dfrac{3}{c}\right)\div\left(1-\dfrac{1}{c^2}\right).$

22. $\left(b+\dfrac{bc}{b-c}\right)\div\left(b-\dfrac{bc}{b+c}\right).$

23. $\dfrac{x}{x+1}+\dfrac{3x}{2+2x}-2.$

24. $\dfrac{3}{x+2}+\dfrac{4}{x+3}=\dfrac{7}{x+6}.$

25. $\left(\dfrac{b}{c}+\dfrac{c}{b}+2\right)\div\left(1+\dfrac{b}{c}\right).$

26. $\dfrac{1}{x}-\dfrac{x+1}{x(x-1)}+\dfrac{1}{x-1}.$

27. $\dfrac{2}{x-1}+\dfrac{1}{2-x}-\dfrac{1}{x-3}.$

$\dfrac{5}{x-2}-\dfrac{3}{x}=\dfrac{12}{x(x-2)}.$

29. $\dfrac{(x+y)^2-z^2}{(x+y-z)^2}.$

30. $\left(a-6+\dfrac{10}{a+5}\right)\left(a-1-\dfrac{6}{a+4}\right)$. 31. $\dfrac{x}{2x-4}-\dfrac{x}{3x-6}=1$.

32. $\{(r+s)^2+(r-s)^2\}\div\left\{\dfrac{r}{s}+\dfrac{s}{r}\right\}$. 33. $\dfrac{1}{x+2}=\dfrac{3}{x-20}-\dfrac{2}{x-5}$.

34. $\dfrac{x+6}{y-2}=\dfrac{2}{3}$, $\dfrac{x+4}{y+1}=\dfrac{4}{7}$. 35. $\dfrac{4(a-b)}{b-1}=2\frac{1}{2}$, $\dfrac{b+1}{5a}=\dfrac{1}{8}$.

[*Note. For additional examples, see Appendix, Ex. S.* 14, *p.* 306.]

Problems

Example 13. From London to Bristol is 120 miles. The average speed of one train is 9 miles an hour more than that of another train and this train takes 40 minutes less time over the journey. Find the speed of each train.

Let the speed of the faster train be x miles an hour.

$\therefore$ the speed of the slower train is $(x-9)$ miles an hour.

Then the faster train goes 120 miles in $\dfrac{120}{x}$ hours, and the slower train goes 120 miles in $\dfrac{120}{x-9}$ hours.

But the time taken by the faster train is 40 min. or $\frac{2}{3}$ hour less than the time taken by the slower train.

$$\therefore \frac{120}{x}+\frac{2}{3}=\frac{120}{x-9};$$

$$\therefore 360(x-9)+2x(x-9)=360x;$$

$$\therefore 360x-9\times 360+2x^2-18x=360x;$$

$$\therefore 2x^2-18x-9\times 360=0; \quad \therefore x^2-9x-9\times 180=0;$$

$$\therefore (x-45)(x+36)=0; \quad \therefore x-45=0 \text{ or } x+36=0;$$

$$\therefore x=45 \text{ or } -36.$$

But the conditions of the problem require that x should be positive; therefore we take $x=45$ and disregard $x=-36$.

Also if $x=45$, $x-9=45-9=36$.

$\therefore$ the faster train averages **45 miles an hour,**
and the slower train averages **36 miles an hour.**

Check: The faster train takes $\frac{120}{45}$ hours $=\frac{8}{3}$ hours $=2$ hr. 40 min.;
The slower train takes $\frac{120}{36}$ hours $=\frac{10}{3}$ hours $=3$ hr. 20 min.;
3 hr. 20 min. $-$ 2 hr. 40 min. $=$ 40 min.

Note. Every value of x which satisfies the data of the problem must satisfy the equation, $\dfrac{120}{x}+\dfrac{2}{3}=\dfrac{120}{x-9}$. The converse is not

necessarily true; it does not follow that every value of x which satisfies the equation must satisfy *all* the data of the problem. Here, there is another condition to be satisfied, namely x must be positive.

EXERCISE XIII. h

1. The bill at a shop for a luncheon party is 24 shillings. If one of the party has no money, the others will have to pay an extra 4d. each. How many were in the party?

2. If the average speed of a train could be increased by 10 miles an hour, a journey of 180 miles would take 54 minutes less time. What is the average speed?

3. The price of eggs having risen by $\frac{1}{2}$d. each, it costs 2d. more to buy 20 eggs than it cost to buy 24 eggs before. What was the former price of an egg?

4. A train running between two towns arrives at its destination 10 minutes late when it averages 48 miles an hour and 16 minutes late when it averages 45 miles an hour. What is the distance between the towns?

5. From London to Crewe is 160 miles. An express train averages 10 miles an hour more than another train and does the journey in 32 minutes less time. What is the average speed of the express?

6. A housekeeper sent 7s. 7d. for the purchase of a certain number of pounds of sugar; the price of sugar had, however, fallen 1d. per lb. so that she received 2 lb. more than she expected and also 3d. in change. Find the price of sugar per lb. at first.

7. When the price of coal rises 8s. a ton, I obtain 5 cwt. less for £2. What was the old price?

8. A wheel, centre C, rests against a vertical step OP and touches the ground at A; OA is level and exceeds OP by 8 inches. If the radius of the wheel is 20 inches, find the height of the step.

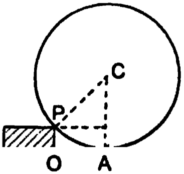


FIG. 189.

9. An aeroplane takes 25 minutes longer to fly 100 miles against a steady wind than with it. The aeroplane travels 100 miles an hour in still air. What is the velocity of the wind?

10. A man can swim 50 yards per minute in still water. He takes $3\frac{3}{4}$ minutes longer to swim 100 yards against a current than with it. What is the speed of the current?

11. A, B, C are 3 points on a circle; the tangent at A cuts BC produced at T; then it can be proved that $\frac{CT}{AT}=\frac{AT}{BT}=\frac{AC}{AB}$. If BC = 9 in., AT = 6 in., AC = 4 in., find the lengths of CT and AB.

12. If an object is at a distance of u feet from a spherical mirror of focal length f feet, the distance of the image from the mirror is v feet where $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$. If the focal length is 4 feet, find, to the nearest inch, the distance of the object from the mirror, if it is 1 yard nearer the mirror than the image is.

13. In a thousand-mile motor race, A's average speed is 10 miles an hour more than B's; and A beats B by 200 miles. What is A's average speed ?

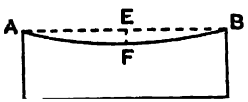
14. A swimming bath is filled by 2 pipes in 12 hours. The smaller pipe by itself takes 7 hours longer than the larger pipe by itself to fill the bath. How long does the larger pipe take to fill the bath ?

15. When the price of petrol is reduced by x per cent., a man uses x per cent. more petrol. His petrol bill is reduced from £50 to £48. What is x ?

16. Two turnstiles A and B admit to a football ground. It takes the average spectator $\frac{1}{5}$ sec. longer to pass through A than through B ; B admits on an average 10 more spectators per minute than A. How many spectators can enter the ground in a quarter of an hour ?

17. A hoop of radius b feet is bowled along with velocity v feet per sec. and comes to a step of height h feet. The hoop will climb the step if v^2 is greater than $\frac{128bh}{(2b-h)^2}$. If the velocity of the hoop is 8 feet per sec. and if its radius is 18 inches, find to the nearest inch the height of the tallest step it can climb.

18. A telegraph wire, of length l feet, connects the tops A, B of two poles, x feet apart, see Fig. 190. If the sag EF at the middle is k feet, then $k=\sqrt{\left\{\frac{3l(l-x)}{8}\right\}}$. Find, to the nearest inch, the length of wire required to join two poles 30 yards apart, allowing for a maximum sag of 2 feet.



x ft.
FIG. 190.

19. A solid rectangular block with square ends is h feet long; its total surface area is S sq. ft. and its volume is V cu. ft. Prove that $(h\mathrm{S}-2\mathrm{V})^2=16h^3\mathrm{V}$. Also find V if $h=2$, $\mathrm{S}=10$. Find also the breadth of the block.

20. A, B, C are 3 points on a circle ; the tangent at A cuts BC produced at T ; then it can be proved that $\frac{\mathrm{AB}^2}{\mathrm{AC}^2}=\frac{\mathrm{BT}}{\mathrm{CT}}$. If AB exceeds AC by 1 inch and if AC = CT and AB = BC, find the length of AB, correct to $\frac{1}{10}$ inch.

[*For a revision exercise on Ch. XI-XIII, see Appendix, Ex. R.* 7, *p.* 269.]

TEST PAPERS B 11-20

B. 11

1. (i) Find the value of $9x^2 - 36x$ when $x = -1\frac{1}{3}$.
   (ii) For what value of x has $9x^2 - 36x$ the same value as it has when $x = -3$?

2. Solve (i) $Q - 1 = \frac{1}{4}P$, $\frac{1}{5}P = \frac{2}{3}Q$;
   (ii) $(7 - x)(2x + 3) = 0$.

3. Simplify (i) $\dfrac{x^2 + 3x - 4}{x - 1}$; (ii) $\dfrac{1}{y^2} + \dfrac{1}{y} - \dfrac{1}{y - 1}$.

4. If $x = 2$, $y = -5$ satisfy both the equations,
$$ax + by + 1 = 0; \qquad (b - 1)x + 5y + 3a = 0,$$
find the values of a, b.

5. A bus from P to Q at 15 m.p.h. takes 24 minutes less than the tram whose speed is 10 m.p.h. How far is P from Q?

B. 12

1. (i) Multiply $x^2 + xy + y^2$ by $x - y$.
   (ii) Simplify $\left(\dfrac{a}{x} - \dfrac{a^2}{x^2}\right) \div \dfrac{a^2}{x}$.

2. Solve (i) $(x - 1)(x - 3) - (x - 5)(x + 2) = 6$;
   (ii) $r^2 = 5r$.

3. Simplify (i) $\dfrac{x + 1}{x^2 - 4x - 5}$; (ii) $\dfrac{a}{ab - b^2} - \dfrac{b}{a^2 - ab}$.

4. Find a pair of numbers, x and y, satisfying $2x - 11y = 4$ and such that one of them is six times the other.

5. The lengths of the sides of a triangle are $x + \frac{1}{2}y$, $y + \frac{1}{3}x$, $5x - 4y + 3$ inches. If the triangle is equilateral, prove that the triangle, whose sides are of lengths xy, $\dfrac{x + 3}{x}$, $\dfrac{y + 4}{y}$ inches, is also equilateral.

B. 13

1. The stretched length, l cm., of a spring supporting a weight, w gm., is given by the formula, $l = 0{\cdot}05w + 15$. What is the natural length of the spring? What load will stretch it to twice its natural length? What increase of load causes an increase in length of 2 cm.?

2. Solve (i) $\dfrac{3r}{4} - \dfrac{s}{3} = 2s - r = 6$.
   (ii) $t^2 + 2t = 15$.

3. Simplify (i) $\dfrac{x^2 - 3x - 10}{xy + 2y}$; (ii) $\dfrac{1}{a - 3b} - \dfrac{5b}{a^2 - ab - 6b^2}$.

4. If $y = mx + c$ where m, c are constants, and if $y = -3$ when $x = 2$, and $y = 5$ when $x = 4$, find y when $x = 5$.

5. Find two consecutive odd numbers such that the sum of their squares is 290.

B. 14

1. (i) Multiply $2x^2 - x - 3$ by $x + 1$.
   (ii) Pick out the coefficients of x^2 and x^3 in
$$(3x^3 - x^2 - 2x + 4)(2x - 1).$$

2. Solve (i) $y = 3{\cdot}4 - 0{\cdot}6x,\ x = 3{\cdot}25 - 0{\cdot}7y$.
   (ii) $2p^2 - p = 3$.

3. Simplify (i) $\dfrac{2x^2 + 5xy - 3y^2}{2x^2 + xy - y^2}$; (ii) $\left(\dfrac{16}{x^2} - 1\right) \times \dfrac{x}{x^2 + 3x - 4}$.

4. For a certain journey, the charge for W lb. of luggage is P pence where $P = \frac{3}{2}\left(\frac{W}{6} - 12\right)$. Interpret this formula in words. [Fractions of a penny, less than 1d., are reckoned as 1d.]
Find how much luggage can be taken for a shilling.

5. The adjacent sides of a rectangle are $2(x - 1)$ inches and $(x + 3)$ inches; those of another rectangle of the same area are $3(x - 2)$ inches and $(x + 1)$ inches. Find their perimeters.

B. 15

1. Draw rough figures to illustrate the expanded values of
(i) $(a + 2)(a + 3)$; (ii) $(b + c + 3)^2$.

2. Solve (i) $\dfrac{p + 1}{q} = \dfrac{2}{3},\ \dfrac{p - 2}{q + 3} = \dfrac{3}{7}$;
   (ii) $t^2 + 2t = 35$.

3. Simplify (i) $\dfrac{1 - a - b + ab}{1 - a + b - ab}$; (ii) $\dfrac{4}{x^2 - 2x - 3} + \dfrac{2}{x^2 + 4x + 3}$.

4. If $x^2 + cx - 12 \equiv (x + a)(x + b)$ where a, b, c are integers, find all the possible values of c.

5. A is 3 times as old as B. In 3 years' time, B will be 3 times as old as C; in 15 years' time, A will be 3 times as old as C. What are their present ages?

B. 16

1. Find x and y if $\dfrac{x}{3} + \dfrac{y}{7} = 1$ and $(x - 3y)(y - 3x) = 0$.

2. Solve (i) $\dfrac{3}{x} - \dfrac{2}{x + 5} - \dfrac{3}{3x - 5}$,
   (ii) $2y^2 + 5y = 12$.

3. Simplify (i) $\dfrac{x-\dfrac{1}{x}}{(1+x)^2}$; (ii) $\dfrac{x-y}{x^2-2xy-3y^2}-\dfrac{x+y}{x^2-4xy+3y^2}$.

4. Draw on the same diagram the graphs of $y=\dfrac{3x}{2}+4$ and $y=6-\dfrac{x}{4}$, and solve graphically these simultaneous equations. Compare by solving algebraically.

5. If F° Fahrenheit is the same temperature as C° Centigrade, $F=a\,.\,C+b$ where a, b are constants. Use the facts that 32° Fahrenheit = 0° Centigrade and 212° Fahrenheit = 100° Centigrade, to find a, b.

Also find t, if $\left(\dfrac{2t}{3}\right)^\circ$ Fahrenheit is the same temperature as $\left(\dfrac{2t}{9}\right)^\circ$ Centigrade.

B. 17

1. (i) Multiply $2-3x-x^2$ by $1+2x$.
   (ii) Pick out the coefficients of x and x^3 in
   $$(4-x-x^2-2x^3)(5x-2).$$

2. Solve (i) $a-5=\frac{1}{7}(b+3)$, $b-12=\frac{1}{5}(4a-2)$;
   (ii) $n+\dfrac{9}{n}=6$.

3. Simplify (i) $\left(x+1-\dfrac{6}{x}\right)\div(3-2x-x^2)$;
   (ii) $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}+\dfrac{4ab}{b^2-a^2}$.

4. Draw the graph of $y=(3-x)(x+2)$ from $x=-3$ to $x=4$.
   Solve graphically (i) $(3-x)(x+2)=5$; (ii) $(3-x)(x+2)=2$;
   (iii) $(3-x)(x+2)+3=0$.

   What is the greatest value of $(3-x)(x+2)$?

5. Find three consecutive numbers such that the square of their sum exceeds the sum of their squares by 382.

B. 18

1. Which of the following are perfect squares, and what are their square roots ?

   (i) x^2-4x+4; (ii) $x^2+8x+64$;
   (iii) $4x^2-6x+9$; (iv) $9x^2+36x+36$

2. Solve (i) $x(x+1)+(x+2)(x+3)=2(x-2)(x+4)$.
   (ii) $6t^2+t=2$.

3. Simplify (i) $\dfrac{2(x-y)+3(x+y)}{5x^2-4xy-y^2}$; (ii) $\dfrac{x+1}{x^2-x}-\dfrac{x+2}{x^2-1}$.

4. (i) What is b if $x-5$ is a factor of $x^2+bx-30$?

(ii) What are b, c if $x+4$ and $x-7$ are both factors of $3x^2+bx+c$?

5. For a run of 60 miles, a train can save 10 minutes if its ordinary speed is increased by 4 miles an hour. What is the ordinary speed ?

B. 19

1. (i) Simplify $\dfrac{cx-dx}{dy-cy}\times\dfrac{y^2}{x^2}$.

(ii) For what value of x is $(3x-7)^2=(3x+8)^2$?

2. Solve (i) $\dfrac{z-1}{z+2}+\dfrac{2z+3}{z-1}=3$.

(ii) $x^2+2x=7$, correct to 1 place of decimals.

3. Simplify (i) $\dfrac{x^2-3x-4}{x^2-3x}\div\dfrac{x^2-4x}{x+3}$;

(ii) $\dfrac{1}{1-x}-\dfrac{x}{(x-1)^2}-1$.

4. Draw the graph of $y=x^2-5x$ from $x=-1$ to $x=6$. Solve graphically the equations, (i) $x^2-5x=3$; (ii) $x^2-5x=-3$; (iii) $x^2-5x+5=0$; (iv) $2x^2-10x+11=0$.

What can you say about c if $x^2-5x+c=0$ has no roots ?

5. When sugar falls in price 1d. per lb., I can buy for 8 shillings six more pounds than I formerly bought for 7s. 6d. What was the old price per lb. ?

B. 20

1. What can you say about the value of x (i) if $(x+4)(y-7)\neq 0$; (ii) if $(2x+1)(y^2+1)=0$; (iii) if $(3x+5)^2+(y-2)^2=0$.

2. Solve (i) $9x+5y+3=\frac{1}{2}(2x-5y-4)=5\frac{1}{2}-y$.

(ii) $x^2-5x=10$, correct to 1 place of decimals.

3. Simplify (i) $\dfrac{x^2+3x-10}{x^2+2x-15}\div\dfrac{x^2+3x+2}{x^2-4x+3}$;

(ii) $\left(1+\dfrac{x}{y}\right)\left(1+\dfrac{y}{x}\right)\div\left(x-\dfrac{y^2}{x}\right)$.

4. If the roots of $ax^2+bx-5=0$ are $2\frac{1}{2}$ and $-\frac{1}{3}$, find the values of a, b.

5. If $4n$ minutes past ten is the same time as n^2 minutes to eleven, find n.

[*For additional test papers on Ch. I-XIII, see Appendix, Q.* 6-10, *p.* 321.]

CHAPTER XIV

LITERAL RELATIONS

Transformation of Formulae

The next Exercise revises and extends previous work.

Example 1. The volume, V cu. in., of a circular cone of base-radius r in. and height h in., is given by the formula,

$$V = \tfrac{1}{3}\pi r^2 h.$$

Make (i) h, (ii) r the subject of the formula.

$$\tfrac{1}{3}\pi r^2 h = V; \qquad \therefore \ \pi r^2 h = 3V;$$

(i) Divide each side by πr^2, $\qquad \therefore \ \mathbf{h} = \dfrac{3\mathbf{V}}{\pi \mathbf{r}^2}$.

(ii) Divide each side by πh, $\qquad \therefore \ r^2 = \dfrac{3V}{\pi h}$;

Take the square root of each side,

$$\therefore \ \mathbf{r} = \sqrt{\left(\frac{3\mathbf{V}}{\pi \mathbf{h}}\right)}.$$

We do not write $r = \pm\sqrt{\left(\dfrac{3V}{\pi h}\right)}$ because r is the radius in inches of the base of the cone.

Example 2. The following equation is used for an experiment in optics,

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{zu}.$$

Make (i) u, (ii) z the subject of the formula.

(i) Multiply each side by $f \cdot u$, $\qquad \therefore \ \dfrac{fu}{f} = \dfrac{fu}{u} + \dfrac{fu}{zu}$;

$$\therefore \ \mathbf{u} = \mathbf{f} + \frac{\mathbf{f}}{\mathbf{z}}.$$

This may also be written, $u = \dfrac{fz + f}{z} = \dfrac{f(z+1)}{z}$.

(ii) Multiply each side by $f \cdot u \cdot z$, $\qquad \therefore \ uz = fz + f$;

$$\therefore \ uz - fz = f; \qquad \therefore \ z(u - f) = f;$$

$$\therefore \ \mathbf{z} = \frac{\mathbf{f}}{\mathbf{u} - \mathbf{f}}.$$

EXERCISE XIV. a

1. If a train, weight W tons, travels round a curve of radius r feet, at v feet per second, the *horizontal* force exerted by the rails on the train is P tons, where $P = \frac{Wv^2}{32r}$. Make (i) r, (ii) v the subject.

2. In Fig. 191, AE bisects $\angle$CAB; if $\angle$DAB $= x°$, $\angle$DAE $= y°$,

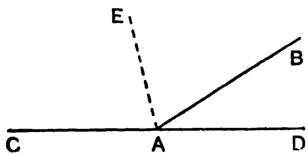


FIG. 191.

find a formula for (i) y in terms of x, (ii) x in terms of y.

3. Interpret the simple interest formula, $A = P + \frac{PRT}{100}$.

Make (i) P, (ii) T the subject of the formula.

Find T if A = 504, P = 420, R = 5.

4. If a car increases its speed from u ft. per sec. to v ft. per sec. in t seconds, at a steady rate of increase, a ft. per sec. every second, then $v = u + at$. Make a the subject.

5. If, with the data of No. 4, the car travels s feet in this time, t seconds, we have the following formulae:

(i) $s = \frac{u+v}{2} \cdot t$; (ii) $s = ut + \frac{1}{2}at^2$;

(iii) $v^2 = u^2 + 2as$; (iv) $s = vt - \frac{1}{2}at^2$.

Use the *appropriate* formula to obtain the following expressions:

(*a*) v in terms of s, u, t; (*b*) v in terms of s, u, a;

(*c*) u in terms of s, v, a; (*d*) a in terms of s, u, t;

(*e*) v in terms of s, a, t; (*f*) a in terms of s, u, v;

(*g*) t in terms of s, u, v; (*h*) t in terms of u, v, a.

6. A man starts with a salary of £P a year for his first year and receives each year an increase of £M a year. Find a formula for the money, £S, he receives for his nth year of service. Then make (i) M, (ii) n the subject.

7. If a force of P lb. is applied for t seconds to a body weighing W lb., the velocity of the body is increased from u ft. per sec. to v ft. per sec., where $Pt = \frac{W}{g}(v - u)$. Express v in terms of P, W, t, u, g. [g is approximately 32.]

8. The refractive index μ of glass is given by the formula, $d = t \cdot \frac{\mu - 1}{\mu}$, see Ex. VIII. *b*, No. 8. Make μ the subject.

9. Fig. 192 represents three arcs, centres A, B, C; $BC = a$, $CA = b$, $AB = c$; the units are inches. Express r in terms of a, b, c.

10. From a masthead, h feet above the surface of the sea, it is possible to see R miles, where $R = \sqrt{\left(\frac{3h}{2}\right)}$. Make h the subject. What is h, if $R = 12$?

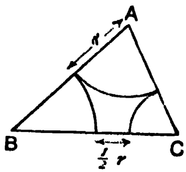


FIG. 192.

11. *Hooke's Law*, see Ch. X, p. 169, is often expressed in the form, $w = \frac{\lambda}{a}(b - a)$; make (i) b, (ii) a the subject.

12. The time, t seconds, of a complete oscillation of a simple pendulum of length l feet is given by the formula, $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$. Make (i) l, (ii) g the subject.

The constant g has the same meaning as in No. 7. Find its value to two significant figures from the fact that a pendulum $3\frac{1}{4}$ feet long makes a *complete* oscillation in 2 seconds; take $\pi = 3 \cdot 14$. This is called a "seconds-pendulum," because the beat is 1 second.

13. A path, x *feet* wide, surrounds a lawn, l yards long, b yards broad; the total area of the path is A sq. feet. Express A in terms of x, l, b. Then make b the subject.

14. Use the fact that the area of a circle, radius r in., is πr^2 sq. in. to interpret the formula, $A = \pi\{(r + t)^2 - r^2\}$.

Make (i) r, (ii) t the subject.

15. If an object is u feet from a spherical mirror of focal length f feet, the distance of the image from the mirror is v feet, where $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$. Make (i) f, (ii) u the subject.

Find u if $f = 4$, $v = 12$.

16. Interpret the formula, $V = \frac{4}{3}\pi r^3$; make r the subject.

17. If V cu. in. of water are poured into a spherical bowl of radius r in., the depth of the water at the centre is h in., where $V = \pi h^2(r - \frac{1}{3}h)$.

(i) Express V in terms of r if $h = 2r$; compare No. 16.

(ii) Express V in terms of r if $h = \frac{1}{2}r$, and in this case make r the subject.

18. Make w the subject of the formula, $E = \frac{wa}{(w + W)b}$.

19. The area of the total surface of a closed circular cylinder, height h in., base-radius r in., is S sq. in.; use Fig. 193, which represents the net area of the surface, to express S in terms of π, r, h. Then make h the subject.

Express r in terms of π, S if $h=\frac{r}{8}$.

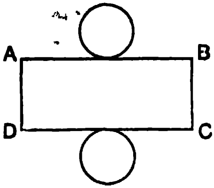


FIG. 193.

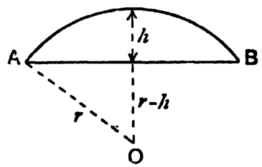


FIG. 194.

20. In Fig. 194, a chord AB of length l inches cuts off from a circle of radius r inches a segment of height h inches. Prove that $l^2+4h^2=8rh$.

Make (i) r, (ii) l the subject. Also find r when $h=5$, $l=12$.

21. If $\frac{a}{x}+\frac{b}{y}=\frac{a}{y}$, express (i) a in terms of b, x, y, (ii) x in terms of a, b, y.

[*Note.* *For additional examples, see Appendix, Ex. S.* 15, *p.* 308.]

Problems involving Letters

Example 3. The length of a field is n times its breadth; the perimeter is p yards; find the breadth.

Let the breadth be x yards.

Then the length is nx yards; $\therefore$ the perimeter is $(2nx+2x)$ yards.

$$\therefore\ 2nx+2x=p; \qquad \therefore\ 2x(n+1)=p;$$

Divide each side by $2(n+1)$, $\qquad \therefore\ x=\frac{p}{2(n+1)}.$

$$\therefore\ \text{the breadth is } \frac{p}{2(n+1)} \text{ yards.}$$

If you do not see how to solve a problem containing letters, invent a similar problem *with numbers*, and work that out first. Then apply the same method to the given problem.

Thus, in the above case, first solve the following:

The length of a field is 5 times its breadth; the perimeter is 400 yards; find the breadth.

EXERCISE XIV. b

1. (i) I share 4 shillings between 2 boys so that one gets 5 times as much as the other. How much does each get ?

(ii) I share p shillings between 2 boys so that one gets n times as much as the other. How much does each get ?

2. (i) The sum of two consecutive integers is 91. What are they ?

(ii) The sum of two consecutive integers is $2N+15$. What are they ?

3. A has £p, B has £q ; how much must A give B so that they have equal amounts ?

4. A man, N years old, has a son, n years old. In how many years' time will the man be (i) just twice as old as his son, (ii) just t times as old as his son ?

5. A hall is k times as long as it is wide ; and its floor area is s sq. feet. What is its length ?

6. One tap supplies per minute k times as much water as another tap ; together, they supply n gallons in t minutes. How many gallons per minute does the first tap supply ?

7. A man rows upstream at u miles an hour and back to the same place at v miles an hour, and takes half an hour altogether. How far upstream did he go ?

8. Share c shillings between 2 boys so that one gets six pence more than twice what the other gets.

9. $N+(r$ per cent. of $N)$ equals k. Find N in terms of r, k.

10. Two rods, of lengths a in. and $(a+b)$ in., are cut down by equal amounts. If the first is then just half the length of the second, how long is each ?

11. A river flows at v miles per hour. What is the speed through the water of a steamer that can go downstream n times as fast as upstream ?

12. If a man sells a bicycle for £P his profit is n times as much as his loss would be if he let it go for £Q. What did the bicycle cost him ?

13. For what value of p will $\dfrac{a+p}{b-p}$ reduce to $\dfrac{4a}{3b}$?

14. If I run to the station at u miles per hour I shall have p minutes to spare ; but if I walk at $\dfrac{3u}{4}$ miles per hour I shall miss the train by np minutes. How far off is the station ?

15. A certain sum of money will pay a man's wages for b days and will pay a boy's wages for c days. For how long will it suffice to pay for both a man and a boy ?

16. If A gives half his money to B, then B will have n times as much as A. If B gives c shillings to A, then A will have twice as much as B. How much has each ?

17. A square lawn has a paved path, d yards wide, all round it. If the area of the path is s sq. yards, find the length of the lawn.

18. For a railway journey, n lb. of luggage is allowed free and the charge for the rest is a penny per w lb. How much luggage do I take, if I pay at the rate of a penny per W lb. on the whole of it ?

Quadratic Equations

Example 4. Solve for x, $ax^2+bx+c=0$.

$$ax^2+bx=-c.$$

Divide each side by a, $\quad \therefore\ x^2+\frac{b}{a}x=-\frac{c}{a}.$

As on p. 205, add to each side $\left(\frac{b}{2a}\right)^2$,

$$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Take the square root of each side. This is possible, if b^2-4ac is a *positive* number, or zero ; *it is impossible, if b^2-4ac is a negative number.*

$$\therefore\ x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a};$$

$$\therefore\ \text{either } x=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a} \quad \text{or} \quad x=-\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$$

This is written more shortly as $x=\dfrac{-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})}}{2\mathbf{a}}$.

And we ought to say as well, "provided that b^2-4ac is not negative."

Solution by Formula

Example 4 gives *a formula* for solving any quadratic equation, which has roots. A simple example of a quadratic, which has no roots, was given on p. 207.

The roots of the quadratic equation,

$$\mathbf{ax^2+bx+c=0}$$

are

$$\mathbf{x}=\frac{-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})}}{2\mathbf{a}},$$

provided that b^2-4ac is not negative.

Example 5. Solve $5x^2 = 9x + 6$, expressing each root correct to two places of decimals.

$$5x^2 - 9x - 6 = 0.$$

The equation, $ax^2 + bx + c = 0$, is equivalent to $5x^2 - 9x - 6 = 0$, if $a = 5$, $b = -9$, $c = -6$.

$$\therefore \text{ the roots are } x = \frac{-(-9) \pm \sqrt{\{(-9)^2 - 4(5)(-6)\}}}{2(5)}$$

$$= \frac{9 \pm \sqrt{(81 + 120)}}{10} = \frac{9 \pm \sqrt{201}}{10};$$

Also $\sqrt{201} \simeq 14 \cdot 18$,

$$\therefore\ x = \frac{9 \pm 14 \cdot 18}{10};$$

$$\therefore\ x = \frac{23 \cdot 18}{10} \text{ or } -\frac{5 \cdot 18}{10}$$

$$= 2 \cdot 318 \text{ or } -0 \cdot 518;$$

$$\therefore\ x = 2 \cdot 32 \text{ or } -0 \cdot 52, \text{ to 2 decimal places.}$$

Do not use the formula, if you can solve the equation by the **direct factor method.**

Do not use the formula, if you can solve the equation *rapidly* by completing the square.

If you use the formula, *substitute first, and simplify afterwards.* Do not try to do both at once.

If you intend to use the formula, you must learn it by heart.

EXERCISE XIV. c

Solve, *when possible,* the following equations, by whatever method is best. *If there are no roots,* say so.

If the roots are irrational, give their values correct to one place of decimals.

1. $x^2 - 3x - 1 = 0$.
2. $x^2 - 3x + 1 = 0$.
3. $x^2 + 5x - 3 = 0$.
4. $x^2 + 7x = 5$.
5. $x^2 - 7x = 18$.
6. $x^2 = 8x + 7$.
7. $2x^2 + 3x = 3$.
8. $2x^2 - 5x = 4$.
9. $2x^2 = 3x + 2$.
10. $3x^2 - 4x + 2 = 0$.
11. $3x^2 + 5 = 8x$.
12. $6x^2 = 2x + 1$.
13. $5x^2 - 7x - 4 = 0$.
14. $2x^2 + 7 = 7x$.
15. $8x^2 - 3x = 11$.
16. $7x^2 + x - 1 = 0$.
17. $10x - 5x^2 = 3$.
18. $6x^2 + 11x = 10$.
19. $\frac{x^2}{3} - \frac{x}{2} = \frac{1}{5}$.
20. $x + \frac{1}{x} = 4$.
21. $\frac{3x^2}{4} - x = \frac{1}{3}$.
22. $\frac{2x}{3} - \frac{5}{2x} = 1\frac{1}{2}$.
23. $3x + \frac{1}{3x} = 1$.
24. $10x + 11 = \frac{6}{x}$.

25. $x^2+2\cdot5x=1\cdot8$. **26.** $2x^2-1\cdot7=1\cdot2x$. **27.** $(x-5)^2=36$.

28. $(x+1)^2=20$. **29.** $(x-3)^2=2x^2-18$. **30.** $x^2+(x-1)^2=0$.

31. Solve (i) by completing the square, (ii) by the formula

$$(a)\ x^2+2kx=l^2\ ;\quad (b)\ px^2+2qx+r=0.$$

32. What is the sum of the roots

$$x=\frac{-b+\sqrt{(b^2-4ac)}}{2a} \text{ or } \frac{-b-\sqrt{(b^2-4ac)}}{2a}\ ?$$

33. (i) What are the roots of $(x-2)^2=7$?

(ii) Combine into a single statement $x-3$ equals $\sqrt{5}$ or $-\sqrt{5}$.

(iii) Combine into a single statement $x=5+\sqrt{2}$ or $5-\sqrt{2}$.

34. Construct an equation whose roots are

(i) $1+\sqrt{3},\ 1-\sqrt{3}$; (ii) $-2+\sqrt{5},\ -2-\sqrt{5}$;

(iii) $\dfrac{5+\sqrt{3}}{2},\ \dfrac{5-\sqrt{3}}{2}$; (iv) $\dfrac{-3+\sqrt{2}}{2},\ \dfrac{-3-\sqrt{2}}{2}$.

35. What can you say about the value of c if $x^2-10x+c=0$ has roots ?

36. What can you say about the value of b if $x^2+2bx+9=0$ has roots ?

37. What can you say about the value of k if $x^2+5x+k=0$ has no roots ?

38. What can you say about the value of p if $x+\dfrac{1}{x}=2p$ has no roots ?

39. Without making a table of values, draw rough figures of the graphs of

(i) $y=(x-1)^2-4=x^2-2x-3$; (ii) $y=(x-1)^2=x^2-2x+1$;

(iii) $y=(x-1)^2+4=x^2-2x+5$.

Which of the following equations have roots ?

(a) $x^2-2x-3=0$; (b) $x^2-2x+1=0$; (c) $x^2-2x+5=0$.

40. Without making a table of values, draw rough figures of the graphs of :

(i) $y=4-(x-1)^2=3+2x-x^2$;

(ii) $y=-(x-1)^2=-1+2x-x^2$;

(iii) $y=-4-(x-1)^2=-5+2x-x^2$.

Which of the following equations have roots ?

(a) $3+2x-x^2=0$; (b) $-1+2x-x^2=0$; (c) $-5+2x-x^2=0$.

[*Note. For additional drill-examples, see Exercise E.P.* 19, *p.* 254.]

CHAPTER XV

FURTHER SIMULTANEOUS EQUATIONS

Simultaneous Equations with three Unknowns

Three equations containing 3 unknowns can be reduced to two equations containing 2 unknowns, by eliminating one unknown.

Example 1. Solve the simultaneous equations,

$$2x + y + z = 2 \quad \ldots\ldots\ldots\ldots \text{(i)}$$
$$3x + 2y + 3z = 3 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$
$$6x + 3y - 2z = 1 \quad \ldots\ldots\ldots\ldots \text{(iii)}$$

Multiply (i) by 3, $6x + 3y + 3z = 6$ .... (iv)

From (ii), $3x + 2y + 3z = 3$.

From (iv) and (ii), by subtraction, $3x + y = 3$ .... (v)

Multiply (i) by 2, $4x + 2y + 2z = 4$ .... (vi)

From (iii), $6x + 3y - 2z = 1$.

From (vi) and (iii), by addition, $10x + 5y = 5$;

Divide each side by 5, $\therefore\ 2x + y = 1$. .... (vii)

[We can now find x and y from (v), (vii).]

From (v) and (vii), by subtraction, $x = 2$.

Put $x = 2$ in (vii), $\therefore\ 4 + y = 1$; $\therefore\ y = -3$.

Put $x = 2$, $y = -3$ in (i), $\therefore\ 4 - 3 + z = 2$;

$\therefore\ z = 2 - 1 = 1$.

$\therefore\ \mathbf{x = 2,\ y = -3,\ z = 1.}$

Check by substituting in (ii) and (iii).

Before starting to solve the equations, look at them carefully and see which is the easiest unknown to eliminate.

Eliminate whichever unknown is easiest.

EXERCISE XV. a

Solve the following simultaneous equations :

1. $x + y + z = 6$,
$2x + 3y + z = 13$,
$x + 2y - z = 5$.

2. $2x - y - z = -5$.
$x - 2y + z = 8$,
$3x + y - 2z = -15$.

3. $3a + b - 2c = 12$.
$5a - b - 3c = 9$,
$4a + b - 5c = 17$.

4. $3x + 2y + z = -4$,
$2x - y + 2z = 11$.
$y + z = 2$.

5. $p + q = 10$,
$q + r = 3$,
$r + p = -1$.

6. $u - v = 2$.
$v - w = -10$,
$2u + 3v + w = -16$.

7. $2a+3b-c=19,$
$3a+4c=3,$
$5a-b+c=20.$

8. $2x-y=7,$
$3y-z=-5,$
$4z-x=5.$

9. $p+2q-r=3,$
$3p-q+2r=0.$
$2p+3q+r=7.$

10. $3x+y+z=x-y-z=y-3z+1=2(x+y)-z.$

11. $\frac{1}{3}(x-y)+z=2,\ \frac{1}{2}(x+z)+y=5,\ \frac{1}{2}(x-z)-\frac{1}{3}(y+z)=1.$

12. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10,\ \frac{2}{x}-\frac{1}{y}+\frac{1}{z}=2,\ \frac{3}{x}+\frac{2}{y}-\frac{3}{z}=7.$

[*Note. For additional drill-examples, see Exercise E.P.* 21, *Nos.* 1-6, *p.* 256.]

Graphical Solutions

Any quadratic equation, which has roots, can be solved by reading off the intersections of a suitable straight line with the graph of $y=x^2$.

Example 2. Solve graphically $5x^2-9x-6=0$.

$$5x^2=9x+6;\qquad \therefore\ x^2=\tfrac{1}{5}(9x+6).$$

$\therefore$ the roots of this equation are the values of x which satisfy the *simultaneous* equations, $y=x^2$, $y=\frac{1}{5}(9x+6)$.

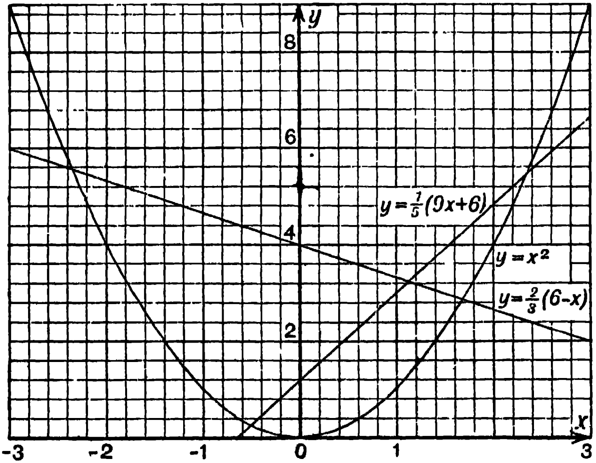


FIG. 195.

The graphs of $y=x^2$ and $y=\frac{1}{5}(9x+6)$ are shown in Fig. 195.

Oral Work on Fig. 195.

(i) Read off the roots of $5x^2 - 9x - 6 = 0$, as accurately as Fig. 195 permits. Compare the answer with Example 5, p. 235.

(ii) Read off the solution of the simultaneous equations,

$$y = x^2, \ 5y = 9x + 6.$$

(iii) Fig. 195 shows the graph of $y = \frac{2}{3}(6 - x)$. What quadratic equation can be solved by using it? What are the roots?

(iv) Solve the simultaneous equations, $y = x^2$, $2x + 3y = 12$.

(v) How can you use the graph of $y = x^2$ in Fig. 195 to solve the equation $x^2 = x + 2$?

By drawing a line *in pencil* or by laying your ruler across the figure, find the roots of $x^2 = x + 2$.

(vi) Repeat (v) for the following:

(*a*) $x^2 + x = 3$; (*b*) $2x^2 + x = 4$; (*c*) $3x^2 = 4x + 12$.

(vii) By drawing a line *in pencil* or by laying your ruler across the paper, show that $x^2 - 2x + 2 = 0$ has no roots.

In Example 2, we have used graphical methods to solve equations which can be solved easily by calculation. There are, however, many types of equations which cannot be solved by any formal method; and if it is necessary to find numerical approximations for the roots, some system of organised guessing must be employed. By drawing suitable graphs, it is often possible to obtain rapidly some idea of the number and value of the roots of equations of this kind.

It is suggested that the following example should be taken orally.

Example 3. Discuss graphically the solution of the equation

$$x^3 + 2x - 5 = 0.$$

First Method. Draw the graph of $y = x^3 + 2x - 5$ and find from it what value, or values, of x make y zero.

(i) What range of values of x is needed for the table? Is it necessary to consider negative values of x? Is it necessary to consider values of x as large as 10? or as large as 5? or as large as 2?

(ii) Make a table of values. Put in one row the values of x^3, in the next row the values of $2x$; use these rows to obtain the values of $x^3 + 2x - 5$.

(iii) Draw the part of the graph which is needed and complete the solution.

Second Method. The equation may be written, $x^3 = 5 - 2x$.

(i) If you draw the graph of $y = x^3$, what other graph must be drawn to solve the given equation ?

(ii) Draw rough figures to show these graphs and then state

(*a*) the number of roots of $x^3 = 5 - 2x$;

(*b*) between what pair of consecutive integers the root, or roots, lie.

(iii) Draw carefully the useful part of the graph $y = x^3$ on as large a scale as possible and then complete the solution.

EXERCISE XV. b

1. Draw the graph of $y = x^2$ for values of x from -3 to $+3$, and use it for the following :

(i) What quadratic can be solved by drawing the graph of $y = 1 - x$ on the same diagram ? What are the roots ?
Solve the simultaneous equations, $y = x^2$, $x + y = 1$.

(ii) Solve graphically $x^2 + x - 2 = 0$.

(iii) Solve graphically $3x^2 - 2x - 6 = 0$.

(iv) What quadratic can be solved by drawing the graph of $y = \frac{4}{5}(x + 7)$ on the same diagram ? What are the roots ?
Solve the simultaneous equations, $y = x^2$, $4x - 5y + 28 = 0$.

(v) Show graphically that $x^2 + x + 1 = 0$ has no roots.

2. Draw the graph of $y = x^3$ for values of x from -3 to $+3$, and use it for the following :

(i) What equation in x can be solved by drawing the graph of $y = 3x + 3$ on the same diagram ? What are the roots ?

(ii) Solve graphically $x^3 = 15 - 5x$.

(iii) What equation in x can be solved by drawing the graph of $y = 5x$ on the same diagram ? What are the roots ?

(iv) Solve graphically $2x^3 - 7x - 4 = 0$.

(v) Solve graphically $3x^3 + 10x + 30 = 0$.

3. With compasses, draw on squared paper a circle, centre the origin, radius 2 inches. Take 1 inch as the unit on each axis.
Show that the coordinates, x and y, of each point on this circle satisfy the equation, $x^2 + y^2 = 4$.
By drawing suitable straight-line graphs, solve the following simultaneous equations :

(*a*) $x^2 + y^2 = 4$, $y = 3(1 - x)$; (*b*) $x^2 + y^2 = 4$, $2y = x + 1$;
(*c*) $x^2 + y^2 = 4$, $2x - 5y = 5$; (*d*) $x^2 + y^2 = 4$, $3x + y = 5$.

Show graphically that the simultaneous equations, $x^2 + y^2 = 4$, $x + y = 3$, have no roots.

4. Solve graphically the simultaneous equations :

(i) $x^2+y^2=25$, $4x+3y=12$; (ii) $x^2+y^2=25$, $4x+3y=25$;
(iii) $x^2+y^2=25$, $3x-4y=18$; (iv) $x^2+y^2=25$, $3x-4y+25=0$.

5. By drawing the graphs of $y=x^3$ and $y=6(x^2-2)$ from $x=-2$ to $x=+2$, solve, as far as the figure allows, these simultaneous equations.

What cubic equation can be solved from these graphs ?

Draw on another diagram the portion of the graphs from $x=5{\cdot}5$ to $x=6$ and then complete the solution.

[*For additional graphical examples, see Appendix, Ex. S.* 10, *p.* 298.]

Simultaneous Linear and Quadratic Equations

Use the linear equation to eliminate one of the unknowns from the quadratic equation.

Example 4. Find x, y from the simultaneous equations,

$$2x-3y=1 \quad \text{.........................(i)}$$

$$2x^2-2xy-3y^2=1 \quad \text{.........................(ii)}$$

From (i), $2x=1+3y$; $\therefore\ x=\dfrac{1+3y}{2}$ ....(iii)

Using (iii), substitute for x in (ii),

$$\therefore\ 2\left(\frac{1+3y}{2}\right)^2-2\left(\frac{1+3y}{2}\right)y-3y^2=1\ ;$$

$$\therefore\ \frac{2(1+6y+9y^2)}{4}-y(1+3y)-3y^2=1\ ;$$

$$\therefore\ 1+6y+9y^2-2y(1+3y)-6y^2=2\ ;$$

$$\therefore\ 1+6y+9y^2-2y-6y^2-6y^2=2\ ;$$

$$\therefore\ -3y^2+4y-1=0\ ; \quad \therefore\ 3y^2-4y+1=0\ ;$$

$$\therefore\ (y-1)(3y-1)=0\ ; \quad \therefore\ y-1=0 \text{ or } 3y-1=0\ ;$$

$$\therefore\ y=1 \text{ or } \tfrac{1}{3}.$$

Substitute in (iii). If $y=1$, $x=\dfrac{1+3}{2}=2$.

If $y=\frac{1}{3}$, $x=\dfrac{1+1}{2}=1$.

$\therefore$ the solution is $\mathbf{x=2,\ y=1}$, *or* $\mathbf{x=1,\ y=\frac{1}{3}}$.

Before starting to solve the equations, look at them carefully and see which is the easier unknown to eliminate.

Eliminate whichever unknown is easier.

When you have found one unknown, the other should be found by substituting in the given *linear* equation, or its equivalent.

EXERCISE XV. c

Solve the following simultaneous equations. In each case, write down beforehand whether it is easier to eliminate x or y or whether it makes no difference.

1. $x=y,\quad 2x^2-y^2=1.$

2. $x+y=7,\quad x^2+y^2=29.$

3. $x-2y=1,\quad xy=3.$

4. $2x+2y=3,\quad x^2-xy=\frac{1}{2}.$

5. $x+y=0,\quad xy-y^2=-8.$

6. $x-y=1,\quad xy-2x=4.$

7. $3x+y=4,\quad xy=1.$

8. $3x-4y=4,\quad y^2=x.$

9. $x+y=6,\quad x^2+2y^2=24.$

10. $x^2-y^2=40,\quad x+y=10.$

11. $2x-y=5,\quad xy=0.$

12. $2x-3y=12,\quad x^2-xy=15.$

13. $x-3y=0,\quad 2x^2+3xy-2x=21.$

14. $3x-y=8,\quad 3x^2-xy+9=y^2.$

15. $x-y=10,\quad \frac{12}{y}-\frac{12}{x}=5.$

16. $x+y=1,\quad \frac{x}{y}+\frac{y}{x}=2\frac{1}{2}.$

17. $3x-2y=7,\quad x^2+xy+y^2=3.$

18. $x-3y+1=0,\quad 3x^2-7xy=5.$

Obtain a linear equation from the pair of given equations (Nos. 19-21), and then solve them.

19. $xy=12,\quad (x-1)(y+2)=15.$

20. $(x+2)(y-2)=144,\quad xy=160.$

21. $xy=6,\quad (x-\frac{1}{2})(y+1)=6.$

22. Solve $2x^2-3y=6$, $3x^2-y^2=11$.

23. Solve $\frac{x}{3}+\frac{y}{4}=5$, $3x^2+xy-y^2=36$.

24. If $x^2-y^2=16$ and $x+y=2$, *write down* (from factors) the value of $x-y$. Then find x and y.

25. If $x^2+y^2=29$ and $x+y=3$, *write down* in succession the values of $x^2+2xy+y^2$; $2xy$; $x^2-2xy+y^2$; $x-y$. Then find x and y.

26. If $x^2+y^2=65$ and $x-y=3$, *write down* in succession the values of $x^2-2xy+y^2$; $2xy$; $x^2+2xy+y^2$; $x+y$. Then find x and y.

[*Note. For additional drill-examples, see Exercise E.P.* 21, *Nos.* 7-26, *p.* 256.]

Simultaneous Quadratic Equations

[*Note.* Ex. XV. d is intended only for those who have been able to work rapidly through Ex. XV. c; otherwise, it should be omitted at a first reading.]

If all terms involving the unknowns are of the second degree, *eliminate the constant term.* This leads to two alternative linear equations; the previous method can then be used.

Example 5. Find x and y from the simultaneous equations:

$$5x^2 - 2xy = 4 \quad \text{.........................(i)}$$

$$9x^2 + 6xy + 4y^2 = 13 \quad \text{.........................(ii)}$$

Multiply each side of (i) by 13, and each side of (ii) by 4,

$$\therefore\ 65x^2 - 26xy = 52 \quad \text{.........................(iii)}$$

and

$$36x^2 + 24xy + 16y^2 = 52 \quad \text{.........................(iv)}$$

From (iii) and (iv), by subtraction, $29x^2 - 50xy - 16y^2 = 0$;

$$\therefore\ (x - 2y)(29x + 8y) = 0;$$

$$\therefore\ \text{either } x - 2y = 0 \text{ or } 29x + 8y = 0 \quad \text{.........................(v)}$$

[It is easier to eliminate y than x, because y appears once only, and to the first degree, in (i).]

From (v), either $2y = x$ or $8y = -29x$;

$$\therefore\ \text{either } y = \frac{x}{2} \text{ or } y = -\frac{29x}{8};$$

(A) If $y = \frac{x}{2}$, from (i)

$$5x^2 - 2x\left(\frac{x}{2}\right) = 4; \quad \therefore\ 5x^2 - x^2 = 4;$$

$$\therefore\ 4x^2 = 4; \quad \therefore\ x^2 = 1; \quad \therefore\ x = \pm 1;$$

$$\therefore\ y = \frac{x}{2} = \pm \tfrac{1}{2}.$$

(B) If $y = -\frac{29x}{8}$, from (i)

$$5x^2 - 2x\left(-\frac{29x}{8}\right) = 4; \quad \therefore\ 5x^2 + \frac{29x^2}{4} = 4;$$

$$\therefore\ 20x^2 + 29x^2 = 16; \quad \therefore\ 49x^2 = 16;$$

$$\therefore\ 7x = \pm 4; \quad \therefore\ x = \pm \tfrac{4}{7};$$

$$\therefore\ y = -\frac{29x}{8} = -\tfrac{29}{8}(\pm \tfrac{4}{7}) = \mp \tfrac{29}{14}.$$

The solution is best given up in tabular form:

$x = 1$	-1	$\tfrac{4}{7}$	$-\tfrac{4}{7}$
$y = \tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{29}{14}$	$\tfrac{29}{14}$

EXERCISE XV. d

Solve the following simultaneous equations :

1. $x^2+2y^2=18$, $xy+y^2=9$.
2. $2x^2+xy=1$, $6xy+4y^2=7$.
3. $9x^2-3xy+y^2=7$, $8xy-y^2=7$.
4. $2x^2+9xy+18y^2=9$, $2xy+9y^2=3$.
5. $3x^2-xy+2y^2=12$, $3x^2-2xy+3y^2=11$.
6. $x^2-5xy+4y^2=10$, $4xy-5x^2=4$.
7. $x^2-5xy+y^2=-5$, $2x^2-9xy+2y^2=-7$.
8. $27y^2-x^2=26$, $6y^2-xy=5$.
9. $3x^2+y^2=21$, $18x^2-9xy-7y^2=63$.
10. $y^2-3xy=19$, $x^2+xy=1\frac{1}{4}$.
11. $x^2-y^2=21$, $x^2-3xy+2y^2=3$.
12. $x^2+3y^2=21$, $2x^2-y^2=14$.
13. $2x^2-xy-3y^2=-7$. $2x^2-5xy+3y^2=-1$.
14. $2x^2-15xy=8$, $20y^2-2xy=21$.
15. $5x^2-5xy+y^2=-1$, $3x^2-3xy+y^2=3$.
16. $x^2-2xy=40$, $2y^2-xy=12$.
17. $4x^2+2y^2=27$, $2xy+y^2=18$.
18. $2x^2-xy+y^2=44$, $x^2+xy+2y^2=16$.
19. $x^2-3xy+y^2=-1$, $x^2-y^2=3$.
20. $x^2-2xy+4y^2=3$, $xy-y^2=\frac{3}{4}$.
21. $x^2-xy-y^2=11$, $2x^2+xy-3y^2=12$.

[*Note. For additional drill-examples, see Exercise E.P.* 21, *Nos.* 27-36, *p.* 256. *For some problems in which* 2 *unknowns may be used, see Appendix, Ex. S.* 16, *p.* 310. *For a revision exercise on Ch. XIV-XV, see Appendix, Ex. R.* 8, *p.* 272.]

TEST PAPERS B. 21-30

B. 21

1. (i) Simplify $\dfrac{c-ac}{e-ae}-\dfrac{d-bd}{e-be}$.

   (ii) Find the L.C.M. of x^2-2x-8, x^2+5x+6, $x^2-8x+16$.

2. Solve (i) $\dfrac{x+1}{\frac{1}{2}}-\dfrac{x-1}{\frac{1}{3}}=\dfrac{x-3}{\frac{1}{4}}$.

   (ii) $x^2+2y^2=17$, $3x^2-5y^2=7$.

3. In Fig. 196, calculate $\angle$PQR.

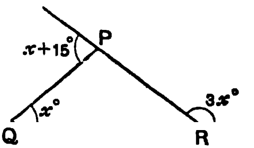


FIG. 196.

4. Prove that the equation $\dfrac{x^2+8x+11}{(x+2)^2}=c$ is satisfied by only one value of x if $c=5$ and if $c=1$. What are the roots if $c=-4$?

5. At a shooting range at a fair, one pays 2d. for a miss and receives 5d. for a hit. A boy has 30 shots and has to pay 11d. How many hits did he score ?

B. 22

1. (i) Simplify $\dfrac{xyz}{xy+yz}$; (ii) Divide x^4-7x^2+1 by x^2-3x+1.

2. Factorise (i) $px-py+qy-qx$; (ii) $(x^2-6x+3)^2-(x-9)^2$.

3. (i) If $x=1\frac{1}{2}$ is one root of $2x^2+x+c=0$, what is the other root and what is the value of c ?

 (ii) Solve $2xy-7x=7$, $3xy-10x=14$.

4. AB is 12 inches long. C is a point on AB such that AC . CB $=35$ sq. inches. Find the length of AC and the area AC^2+CB^2.

5. The hot water tap takes 4 minutes longer than the cold water tap to fill a bath. Both taps together take $3\frac{3}{4}$ minutes. How long does the hot water tap take by itself ?

B. 23

1. (i) One factor of $24x^3-50x^2y-3xy^2+36y^3$ is $2x-3y$, find the others.

 (ii) Factorise $1-(a-b)^2$.

2. Simplify (i) $(y+z)(y-z)+(z+x)(z-x)-(y+x)(y-x)$.

 (ii) $\left(a-3-\dfrac{10}{a}\right)\left(a+1-\dfrac{12}{a}\right)\div\left(a-1-\dfrac{20}{a}\right)$.

3. (i) What is b if $x-5$ is a factor of $x^2+bx-30$?

 (ii) Solve $\dfrac{x}{y}+3=7y$, $(y-1)^2-x=0$.

4. In Fig. 197, both arrows point East. Find x.

5. A stone projected vertically upwards is s feet above the ground after t seconds, where $s = 104t - 16t^2$. How long is it in the air? If it takes the same time to go up as to come down, what is the greatest height it reaches?

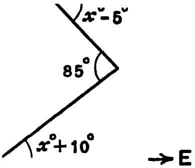


FIG. 197.

B. 24

1. (i) Find the L.C.M. of $x^2 - y^2$, $x^2 + 2xy + y^2$, $x^2 - xy$.

(ii) Divide $a^4 + 4b^4$ by $a^2 - 2ab + 2b^2$.

2. Factorise (i) $a^2 - 4(b-c)^2$;
(ii) $(a+3)(a-1) - (2a-3)(a+1)$.

3. Solve (i) $(3x+4)^2 = 1$;
(ii) $x - 2y = 9$, $y + 3z = 7$, $7z - 3x = 6$.

4. A flat ring is bounded by two concentric circles. The radius of the inner ring is $(r - d)$ inches and the breadth of the ring is $2d$ inches. If the area of the ring is A sq. inches, find r in terms of π, A, d.

5. A woman buys equal quantities of tea at 2s. 6d. and 2s. per lb. If she had divided her money equally between the two kinds, she would have bought altogether half a pound more. How much money does she spend?

B. 25

1. (i) Factorise $a^2 - 7ab - 18b^2$.

(ii) Simplify $\left(x - \frac{4}{x}\right) \div \left(\frac{1}{2} + \frac{1}{x}\right)$.

2. Solve (i) $2x^2 - x = 10$; (ii) $x^2 - 2y^2 = 3x + 4y = 1$.

3. If x shilling per cwt. is the same rate of cost as f farthings per lb., express x in terms of f.

4. In Fig. 198, the arrows point respectively due East and due West. Find x.

5. 100 cu. ft. of a mixed gas weigh 8 lb. The two gases which are mixed weigh respectively 0·077 lb. per cu. ft. and 0·087 lb. per cu. ft. How many cu. ft. of each gas go to form the mixture?

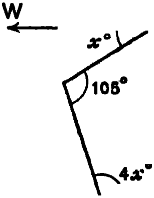


FIG. 198.

B. 26

1. (i) Find the square root of $x^2 - x + \frac{1}{4}$.

(ii) Simplify $\dfrac{1}{3a-3b} + \dfrac{1}{4b-4a}$.

2. Factorise (i) $a^2 - 4b^2c^2$; (ii) $1 + 2y - 15y^2$.

3. Solve (i) $12(x^2 - 1) = 7x$;

(ii) $(x+2)(x-5) = (y+2)(y-5)$, $x - y = 6$.

4. Draw the graph of $y = \frac{1}{2}(x-3)(x+1)$ from $x = -3$ to $x = 5$. Use it to solve the equations:

(i) $(x-3)(x+1) = 2$; (ii) $(x-3)(x+1) = -x$.

What can you say about the value of c if $(x-3)(x+1) = c$ has no roots ?

5. The resistance, R lb. per ton, to the motion of a train travelling at v miles per hour is given by the formula, $R = a + bv^2$, where a, b are constants. When $v = 20$, $R = 9{\cdot}6$, and when $v = 40$, $R = 20{\cdot}4$. Find a, b. Find, correct to 2 significant figures, the value of R when $v = 35$.

B. 27

1. Factorise (i) $6x^2 + xy - 15y^2$; (ii) $a(x-a)^2 + b(a^2 - x^2)$.

2. (i) Write down the square of $3ab - bc$.

(ii) Simplify $\left\{\dfrac{x}{x-y} + \dfrac{x}{x+y}\right\} \div \left\{\dfrac{y}{x-y} + \dfrac{y^2}{(x-y)^2}\right\}$.

3. Solve (i) $6t^2 + t = 15$;

(ii) $2x^2 + 5xy = 2x - 3y = -12$.

4. W lb. of guncotton will demolish a wall h ft. high, t ft. thick, per foot length, if $W = cht^2$, where c is a constant. If 45 lb. of guncotton is needed per foot length for a wall 10 ft. high, 3 ft. thick, how much is needed for a similar wall 12 ft. high, 5 ft. thick, 80 ft. long ?

5. ABC is a triangle such that AB = 7 in., BC = 3 in. If a line PN parallel to AB cuts CA, CB at P, N respectively, it is known that $\dfrac{PN}{NC} = \dfrac{AB}{BC}$. Find CN, if PN = NB.

B. 28

1. Factorise (i) $1 + 4c - 5c^2$; (ii) $cx - dx + dx^2 - cx^2$.

2. Which of the following equations have no roots ?

(i) $x^2 - 5x - 1 = 0$; (ii) $x^2 - 5x + 25 = 0$;

(iii) $x^2 - 5x + 6 = 0$; (iv) $x^2 - 5x + 7 = 0$.

3. Solve (i) $9x^2 = 21x + 44$;

(ii) $2x - y + 3z = 4$, $3x + 5y = 1$, $x + 2y + 2z = 9$.

4. Draw the graph of $y=x^3$ from $x=-2$ to $x=+2$.
Solve graphically the equations, (i) $x^3=3x$; (ii) $x^3-3x+1=0$.

5. A man has to travel 50 miles in 4 hours. He does it by walking the first 7 miles at x m p.h., bicycling the next 7 miles at $4x$ m.p.h. and motoring the remainder at $(6x+3)$ m.p.h. Find x.

B. 29

1. Simplify (i) $\dfrac{x^2-4x-12}{x^2-4}$;

(ii) $\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)\div\dfrac{a^2+b^2}{a-b}$.

2. Make r the subject of the formula, $s=\dfrac{a}{1-r}$.

3. Solve (i) $15x^2+17x=42$;

(ii) $x^2+3xy=4$, $2x^2+5xy=15$.

4. In Fig. 199, calculate the value of x.

5. When petrol costs 3s. 6d. a gallon and oil 4s. a quart, a cyclist finds that the cost of petrol and oil is £4. 10s. per 1000 miles. When the price is reduced to 3s. and 3s. 6d. respectively the cost is £3. 17s. 6d. per 1000 miles. How many miles does he run on a gallon of petrol?

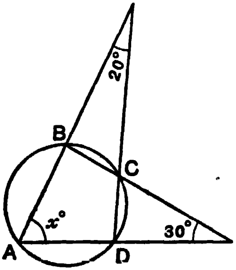


FIG. 199.

B. 30

1. Factorise (i) $4x^2-36y^2$;

(ii) $6x^3-11x^2+4x$.

2. (i) Pick out the coefficients of x and x^3 in

$(4-2x+x^2-3x^3)(1+5x-x^2-2x^3)$.

(ii) Simplify $\left(1+\dfrac{a}{b}\right)\div\left(1-\dfrac{a^2}{b^2}\right)$.

3. Solve (i) $2x+3y=8$, $2x^2=y$;

(ii) $x-x^2=25x-25$.

4. The base of a box is a square of side x feet and the length of cord (exclusive of knots) needed for binding it once round each way is 40 feet. Show that the volume of the box is $2x^2(5-x)$ cu. feet. Represent this function by a graph, from $x=0$ to $x=5$. For what value of x is the volume greatest?
Solve graphically (i) $2x^2(5-x)=25$; (ii) $5x^2-x^3=15$.

5. The perimeter of a triangle is 4 feet and one side is twice another side. If the shortest side is l inches long, prove that $12>l>8$.

[*For additional test papers on Ch. I-XV, see Appendix, Q.* 11-20, *p.* 323.]

EXTRA PRACTICE EXERCISES FOR PART II

EXERCISE E.P. 14

Simultaneous Equations (Chapter IX)

Solve the following pairs of simultaneous equations :

1. $a+b=13$, $a-b=4$.
2. $u-v=11$, $u+v=1$.
3. $x-y=9$, $x+y=0$.
4. $p+q=8$, $q-p=3$.
5. $2l+m=12$, $l=m$.
6. $2u-v=7$, $u+v=8$.
7. $3x-2y=6$, $x+3y=13$.
8. $7x-y=2$, $6x=y$.
9. $2x+5y=8$, $3x+4y=5$.
10. $y=2x+1$, $3y=5(x+1)$.
11. $3x+4y=7$, $y=0$.
12. $4x+3y=1$, $5x+4y=2$.
13. $l+m=2\frac{1}{3}$, $m-l=1\frac{1}{6}$.
14. $a+2b=5\frac{1}{4}$, $a-b=1\frac{1}{2}$.
15. $4x-3y=3$, $y=2x$.
16. $t+2z=12$, $z=4t$.
17. $c=3-2d$, $d=12-2c$.
18. $3p+4q-3=0$, $2p-5q+5=0$.
19. $3x+5y=21$, $x+2y=7$.
20. $2x+7y-22=0$, $3x+6y-15=0$.
21. $4x+3y+13=0$, $3x+2y+8=0$.
22. $2r+s+10=0$, $3r-2s+1=0$.
23. $3p+2z=4$, $4p+z+3=0$.
24. $2x+11y=12$, $3x-2y+19=0$.
25. $c-7=d+2=5$.
26. $2p-1=0=3+q$.
27. $x-3y-5=2x+y-3=0$.
28. $y=2(x+y-1)=-2(x+y+3)$.
29. $2x+3y=x+1=4-y$.
30. $x+2y+3=4x+4y-1=3x+3y+2$.
31. $\frac{1}{2}p+\frac{1}{3}q=1$, $2p+3q+1=0$.
32. $\dfrac{x-1}{3}+\dfrac{y+1}{2}=1$, $\dfrac{2x+1}{5}-\dfrac{3y+1}{4}=5$.
33. $\dfrac{1}{x}-\dfrac{1}{y}=2$, $\dfrac{1}{x}+\dfrac{1}{y}=8$.
34. $\dfrac{3}{y}-\dfrac{1}{z}=1$, $\dfrac{5}{y}+\dfrac{2}{z}=20$.
35. $4x-2y=4y-5x=x+y-3$.
36. $2x-y-3=3y-x+4=5x-8y-4$.

37. $\frac{2l-m}{3}=\frac{l-3m}{2}$,
$\frac{l}{3}=1-m$.

38. $\frac{r-2s}{r}=\frac{3s}{2r}$,
$\frac{3}{r}-\frac{2}{s}=\frac{1}{rs}$.

39. $\frac{x-y}{x+y}=\frac{3}{4}=\frac{x-1}{y+2}$.

40. $\frac{3x-1}{2y+1}=\frac{2x+2}{y+2}=4$.

EXERCISE E.P. 15

Problems involving Two Unknowns (Chapter IX)

1. A knife and fork cost 6s.; the knife costs six pence more than the fork. What is the cost of each?

2. A man distributes x oranges among n boys; if each receives 10 oranges, there are 3 left over; but there are four oranges too few for each boy to receive 11 oranges. Find x and n.

3. I am thinking of two numbers which when added make 90 and are such that one-third of the smaller is equal to one-seventh of the larger. What are they?

4. 3 lb. of jam and 2 lb. of butter cost 8s.; also 6 lb. of jam and 3 lb. of butter cost 14s.; find the cost of 1 lb. of jam and of 1 lb. of butter.

5. 3 cows and 4 sheep cost £52; also 4 cows and 6 sheep cost £71; find the cost of one cow and of one sheep.

6. Can you find two numbers such that three times the smaller exceeds twice the larger by 3, and seven times the smaller exceeds five times the larger by 2?

7. A boy spends 4s. 6d., partly on the entrance money for an exhibition, and the rest on amusements inside; his brother, who spends twice as much on amusements, spends 7s. altogether. What did each spend on amusements and what was the entrance money?

8. In 4 years' time, a father will be 3 times the age of his son; 4 years ago he was 5 times the age of his son. What are their present ages?

9. I have to pay for 500 cigarettes two shillings more than for 3 lb. of tobacco: and I have to pay for 4 lb. of tobacco four shillings more than for 600 cigarettes. What do I pay for one lb. of tobacco and for 100 cigarettes?

10. I wish to give one shilling each to some children, but find I have 1s. 6d. too little to do so. I therefore give them 10d. each and have one shilling over. How many children are there and how much money have I got with me?

11. A owes £4, B owes £5; A could just pay his debt if he borrowed from B one-eighth of what B has; B could just pay his debt if he borrowed from A two-sevenths of what A has. How much has each?

12. Find two numbers such that the first is greater than half the second by 4, and three times the first is less than twice the second by 1.

13. If A gives B two pence, B has four times as much as A. If C gives A and B ten pence each, then B has twice as much as A. How much have A and B ?

14. A heap of half-crowns and florins is worth £2. 10s. ; another heap, which contains only half as many half-crowns but twice as many florins is worth £2. 15s. How many coins of each kind are there in the first heap ?

15. Can you discover a fraction such that the result of either adding 1 to the numerator or of subtracting 3 from the denominator is equivalent to $\frac{1}{3}$? Is there more than one answer ?

16. A and B are two jugs ; an empty pail is just filled by 5 jug-fulls from A and one from B or by 8 jug-fulls from B. If 3 jug-fulls from both A and B are put into it, there is still room for another 2 pints. How much does each jug hold ?

17. Two numbers, x and y, are in the ratio 7 : 5 ; twice the smaller exceeds the larger by 36. What are they ?

18. A shilling weighs $\frac{1}{5}$ oz. and a penny weighs $\frac{1}{3}$ oz. ; 11 coins, some shillings, the rest pence, weigh 2·6 oz. What is the value of the coins ?

EXERCISE E.P. 16

Products (Chapter XI)

Expand the following expressions :

1. $(x-3)(y-3)$.	2. $(r+4)(s-3)$.	3. $(2+c)(3+d)$.
4. $(4-y)(2+z)$.	5. $(1-a)(1-b)$.	6. $(3-a)(b-2)$.
7. $(r+1)(r-1)$.	8. $(x+7)(x+7)$.	9. $(y-5)(y-5)$.
10. $(b-6)(b+6)$.	11. $(c+9)(c-6)$.	12. $(8-n)(3+n)$.
13. $(d-3)(d-3)$.	14. $(6-a)(6+a)$.	15. $(p-8)(p+3)$.
16. $(a+2b)(a+5b)$.	17. $(c-3d)(c-7d)$.	18. $(2y+z)(5y-z)$.
19. $(4x+y)(x-3y)$.	20. $(p+q)(6p-q)$.	21. $(r-s)(3r+s)$.
22. $(2c+1)^2$.	23. $(3d-1)^2$.	24. $(4c-3d)^2$.
25. $(3+2t)(3-2t)$.	26. $(3+2t)^2$.	27. $(\frac{1}{2}a-\frac{1}{3}b)^2$.
28. $(2x-5y)^2$.	29. $(r-3s)(3r+s)$.	30. $(t-1)(1-t)$.

What is the coefficient of x^2 and the constant term in the following products ?

31. $(x-3)(x+5)$.	32. $(2x+1)(7x-1)$.	33. $(1-4x)(1+6x)$.
34. $(2x-5)(3x-2)$.	35. $(3-7x)(2+5x)$.	36. $(5x-7)(4-3x)$.
37. $(3x-1)(2x^2-x-4)$.	38. $(3+2x-5x^2)(4+3x)$.	

39. $(2x+7)(4x-3x^2-2)$.

40. $(6x-x^2-1)(6+x)$.

41. Multiply y^2+2y+1 by $y+1$.

42. Multiply z^2+z+1 by $z-1$.

43. Multiply $2a^2-3a-5$ by $3a+2$.

44. Multiply $3b^2-2bc-c^2$ by $2b-3c$.

45. Multiply $2-7x-5x^2$ by $4+3x$.

46. Multiply $y^2-3yz+9z^2$ by $y-3z$.

47. Multiply $4x-2x^2-3$ by $2x+5$.

48. Multiply $3ab-2a^2-4b^2$ by $3b+2a$.

EXERCISE E.P. 17

Factors (Chapter XI)

Factorise the following :

1. $ab-bc-ad+cd$.
2. $t^2-7t+12$.
3. $49-k^2$.
4. p^2-3p-4.
5. $ax-2bx+ay-2by$.
6. $4c^2-25d^2$.
7. $x^2+4x-21$.
8. $y^2+10y+24$.
9. $ab-2ac+2bd-4cd$.
10. $3p^2+5p-2$.
11. $1-11t+24t^2$.
12. a^4-b^2.
13. $b^2-bc+bd-cd$.
14. $z^2-5z-14$.
15. $(a+2b)^2-4c^2$.
16. $2n^2+n-6$.
17. $xy-xz-y^2+yz$.
18. $1-2t-15t^2$.
19. $12c^2-c-6$.
20. $1+k+k^2+k^3$.
21. $10d^2-3d-1$.
22. $10l^2-19lm+6m^2$.
23. $x^2-xy+yz-xz$.
24. $p^2-(r-s)^2$.
25. $3t^2+9t-30$.
26. $1-m-12m^2$.
27. $1+yz-y-z$.
28. $a^2-16a+48$.
29. $x^2-4(y+z)^2$.
30. $20b^2-49b+30$.
31. $a(a-1)+a-1$.
32. $4t^2+12t-72$.
33. $15+14p-8p^2$.
34. $1-p-q(p-1)$.
35. $12y^2-7y+1$.
36. $12x^2-16x-3$.
37. $4ax-6bx+9by-6ay$.
38. $3x^2+xy-2y^2$.
39. $6x^2-2xy+ay-3ax$.
40. $12r^2+32rs+5s^2$.
41. $x^2+(a+b)x+ab$.
42. $28a^2-3ab-18b^2$.
43. $3x^3-5x^2y+6xy^2-10y^3$.
44. $21c^2+34cd+8d^2$.
45. $20y^2+21yz-54z^2$.
46. $x^2-(p-q)x-pq$.
47. $1+x^2+xy+x^3y$.
48. $2a^4+3a^2bc-2b^2c^2$.
49. $ax+4-2x-2a$.
50. $2x^4+11x^2+15$.
51. $(2a+b)(c+d)+c+d$.
52. $3-5t^2-12t^4$.

53. $x - y + (y - x)^2$.
54. $36a^2 + 27ab - 28b^2$.
55. $105 - 17p - 12p^2$.
56. $(c - d)^2 + c^2 - cd$.
57. $(x + 1)^2 - (x + 1)(2x - 3)$.
58. $x(x + 4y) - 3y(3x - 2y)$.
59. $12x^2 + 5xy - 72y^2$.
60. $(x - y)^2 - 2x(x - y)$.
61. $a(x + y) - xy - a^2$.
62. $x(x + 1) + (x + 4)(x - 3)$.
63. $a(b^2 + c^2) - bc(1 + a^2)$.
64. $x^2y^2 - 7xyz - 18z^2$.
65. $14a^2 - 13ab - 12b^2$.
66. $6x^2 - 4xy + 10yz - 15xz$.

EXERCISE E.P. 18

Quadratic Equations (Chapter XII)

1. If $(x - 3)(y - 5) = 0$, can you find the value of x when (i) $y = 7$; (ii) $y = 5$; (iii) $y = 0$? If so, what is it?

2. If $(a - b)(x + 2) = 0$, find, when possible, the value of x when (i) $a = 3, b = 1$; (ii) $a = 0, b = 1$; (iii) $a = 4, b = 4$.

3. Do you know anything about the numerical value of x if $xy = 10$?

4. What can you say about the values of x and y if

$$(x - 3)(y - 5) = 0\ ?$$

Express the following facts by equations:

5. Either $x = 3$ or $x = 6$.
6. Either $y = 2$ or $y = 5$.
7. Either $t = 2$ or $t = -2$.
8. Either $z = 0$ or $z = 4$.
9. Either $x = 1$ or $y = 1$.
10. Either $x = -2$ or $x = 3$.
11. Either $p = -4$ or $p = -7$.
12. Either $y = 3$ or $z = -2$.
13. Either $x = \frac{1}{2}$ or $x = 3$.
14. Either $t = \frac{2}{3}$ or $t = \frac{4}{7}$.
15. Either $y = 0$ or $y = -\frac{3}{4}$.
16. Either $z = -2\frac{1}{2}$ or $z = -1\frac{1}{3}$.
17. $x = \pm 5$.
18. $s = \pm \frac{2}{3}$.
19. $y = 0$ or 2 or 5.
20. $z = 1$ or -1 or -2.
21. $t = \frac{1}{2}$ or -2 or $1\frac{1}{4}$.
22. $v = 0$ or $\pm \frac{3}{4}$.
23. $x = -\frac{2}{5}$ or $2\frac{1}{2}$.
24. $y = 1$ or $-1\frac{1}{2}$ or $-\frac{1}{3}$.

Solve the following equations:

25. $x(x - 3) = 0$.
26. $(x - 2)(x + 3) = 0$.
27. $(x - 4)^2 = 0$.
28. $3x(x + 2) = 0$.
29. $(2x - 3)(x - 5) = 0$.
30. $(3x + 1)(x + 3) = 0$.
31. $5x(5x - 2) = 0$.
32. $(2x + 3)^2 = 0$.
33. $x(x + 1)(x - 2) = 0$.
34. $(2x - 7)(7x + 2) = 0$.
35. $(x - 4)^2(x + 4) = 0$.
36. $(3 - 2x)(5 + x) = 0$.
37. $(4 + x)(1 - 4x) = 0$.
38. $(1 - x)(2 - x)(3 + x) = 0$.
39. $x^2 - 8x + 15 = 0$.
40. $x^2 - 8x = 0$.
41. $x^2 + 11x + 28 = 0$.
42. $x^2 + 11x = 0$.

43. $x^2+5x-6=0$.
44. $x^2-3x-70=0$.
45. $x^2-6x+9=0$.
46. $x^2+12x+36=0$.
47. $x^2+9x=36$.
48. $x^2-11x=60$.
49. $x(x-1)=72$.
50. $x(x+2)=99$.
51. $2x^2-11x+5=0$.
52. $6x^2=x+2$.
53. $x^2=25$.
54. $4x^2=9$.
55. $15x^2+2x=8$.
56. $14x^2=17x+6$.
57. $(2x+3)^2=25$.
58. $20x^2-7x=6$.
59. $4x^2=7x$.
60. $10x^2+33x+20=0$.
61. $(x+1)(x-3)=12$.
62. $(2x-1)(1+3x)=4$.
63. $x(x-1)=3(x-1)$.
64. $(x+2)^2=5(x+2)$.

EXERCISE E.P. 19

Quadratic Equations (Chapter XII)

In solving the following equations, use the direct factor method whenever it is easier to do so. If the roots are not rational, give each root correct to one place of decimals.

1. $x^2-4x=3$.
2. $x^2+6x=8$.
3. $x^2-5x=7$.
4. $x^2+3x=5$.
5. $x^2+8x+14=0$.
6. $x^2-10x+12=0$.
7. $x^2-7x=18$.
8. $x^2+5x-7=0$.
9. $x^2+x-3=0$.
10. $2x^2-12x=15$.
11. $2x^2+3x=4$.
12. $3x^2-2x=1$.
13. $3x^2-2x=2$.
14. $5x^2-8x+2=0$.
15. $4x^2-3x=3$.
16. $x+\dfrac{1}{x}=3$.
17. $x+5=\dfrac{2}{x}$.
18. $2x+\dfrac{1}{x}=7$.
19. $2x^2+5x=3$.
20. $3x^2+4x=8$.
21. $(x+1)(5x-1)=6$.

Which of the following equations have no roots ? Do not solve any of them.

22. (i) $4x^2-7=0$; (ii) $4x^2+25=0$; (iii) $-x^2=-6$.
23. (i) $x^2-4x-5=0$; (ii) $x^2-4x+4=0$; (iii) $x^2-4x+5=0$;
24. (i) $x^2+10x-1=0$; (ii) $x^2+10x+21=0$; (iii) $x^2+10x+25=0$; (iv) $x^2+10x+26=0$; (v) $x^2-10x+26=0$.

EXERCISE E.P. 20

Fractions (Chapter XIII)

Simplify the following expressions :

1. $\dfrac{3x-3y}{3x+3y}$.
2. $\dfrac{2a+2b}{5a+5b}$.
3. $\dfrac{a^2-ax}{ax-x^2}$.
4. $\dfrac{c+c^2}{3+3c}$.
6. $\dfrac{x^2-xy}{x^2-y^2}$.

7. $\dfrac{ab^2+a^2b}{abc}$. 8. $\dfrac{2x-2y}{(x-y)^2}$. 9. $\dfrac{4x-4y}{6y-6x}$.

10. $\dfrac{a^2-b^2}{(a-b)^2}$. 11. $\dfrac{6x^2-6xy}{4xy}$. 12. $\dfrac{3x-3}{(1-x)^2}$.

13. $\dfrac{b^3-b^2c}{b^2c-bc^2}$. 14. $\dfrac{(a-2)^2}{(2a-4)^2}$. 15. $\dfrac{2x-2y}{3x^2y-3xy^2}$.

16. $\dfrac{x-3}{x^2-x-6}$. 17. $\dfrac{a^2+a-6}{2a+6}$. 18. $\dfrac{ab-b^2}{a^2+ab-2b^2}$.

19. $\dfrac{x^2+6x+8}{x^2-4}$. 20. $\dfrac{x^2+xy-2y^2}{x^2-2xy+y^2}$. 21. $\dfrac{x^2-8x+15}{x^2+2x-35}$.

22. $\dfrac{y^2-2y+1}{y^2+3y-4}$. 23. $\dfrac{x^2-x}{(2x-2)^2}$. 24. $\dfrac{x^2+x-2}{x^2-1}$.

25. $\dfrac{x^2-1}{x}\times\dfrac{x^2}{x^2+2x+1}$. 26. $\dfrac{a^2+a-2}{a^2-a-6}\times\dfrac{a-3}{a-2}$.

27. $\dfrac{x^2+xy}{xy-xz}\times\dfrac{y^2-yz}{xz+yz}$. 28. $\dfrac{4x-12}{3x}\times\dfrac{9x^2}{6x-18}$.

29. $\dfrac{2x}{3x-3}\div\dfrac{4y}{x^2-x}$. 30. $\dfrac{5a}{a-3b}\div\dfrac{2a}{3a-b}$.

31. $\dfrac{x^2-x}{ax-a}\div\dfrac{ax}{ax-x}$. 32. $\dfrac{x^2-xy}{xy+y^2}\div\dfrac{xy-y^2}{x^2+xy}$.

33. $\dfrac{x^2-2xy+y^2}{x^2-y^2}\times\dfrac{x^2+2xy+y^2}{x^2+xy}$. 34. $\dfrac{b^2-c^2}{b^2-2bc+c^2}\div\dfrac{c}{b^2-bc}$.

35. $\dfrac{b^2+2bc+c^2}{b^2+2bc}\times\dfrac{b}{b^2-c^2}$. 36. $\dfrac{a^2-4ab+4b^2}{a^2+ab-6b^2}\div\dfrac{a^2+3ab}{a^2+6ab+9b^2}$.

37. $\dfrac{x}{x-2}-\dfrac{2}{x+2}$. 38. $\dfrac{2x-3}{3x-9}-\dfrac{x-2}{2x-6}$.

39. $\dfrac{1}{3a-3b}+\dfrac{1}{6a-6b}$. 40. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$.

41. $\dfrac{4}{x+3}+\dfrac{12}{x^2-9}$. 42. $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

43. $\dfrac{x+2}{x+1}-\dfrac{x+1}{x+2}$. 44. $\dfrac{ab}{a^2-b^2}-\dfrac{b}{a+b}$.

45. $\dfrac{1}{x^2-4}-\dfrac{1}{x^2+2x-8}$. 46. $\dfrac{a}{a-b}+\dfrac{a}{b-a}$.

47. $\dfrac{3}{x-2}-\dfrac{6}{x^2-4}$. 48. $\dfrac{a+2b}{a^2+ab}+\dfrac{a-b}{ab+b^2}$.

49. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x+y}{x-y}$. 50. $\dfrac{x}{x^2-4x+3}-\dfrac{2}{x^2-x-6}$.

51. $\dfrac{2b}{b^2-a^2}+\dfrac{1}{a-b}$. 52. $\dfrac{2b}{b+c}+\dfrac{c}{b-c}-\dfrac{2b^2}{b^2-c^2}$.

EXERCISE E.P. 21

Simultaneous Equations (Chapter XV)

Solve the following simultaneous equations :

1. $2x-5y-z=-4$, $x+2y+2z=5$, $3x-y-z=-10$.

2. $3x-5y=11$, $2y+3z=34$, $5z-4x=22$.

3. $\frac{1}{2}x-\frac{1}{3}y+z=8$, $4y-7z=-20$, $x+\frac{2y}{3}-\frac{1}{2}z=8$.

4. $2x-y-z=z-x-\frac{1}{2}y=\frac{1}{4}(x+z)-y=\frac{1}{3}(y+z)-2$.

5. $\frac{1}{x}-\frac{1}{y}=4$, $\frac{1}{y}-\frac{1}{z}=5$, $\frac{1}{x}+\frac{1}{z}=1$.

6. $\frac{2}{x}-\frac{3}{y}+4z=16$, $\frac{1}{x}+\frac{6}{y}-z=7$, $\frac{3}{x}-\frac{1}{y}+\frac{1}{3}z=15$.

7. $2x+y=1$, $x^2-2xy=39$.

8. $x-2y=1$, $xy=1$.

9. $x^2+y^2=34$, $x-y=8$.

10. $2x^2+y^2=3$, $x+y=2$.

11. $7x-y=2$; $x^2+y^2=2\frac{1}{2}$.

12. $x^2+y^2=-6x$, $y-2x=3$.

13. $x^2+y^2=x+y=2$.

14. $2y=7x-5xy=3x-xy$.

15. $xy+2x-y=3$, $x+y=1$.

16. $7xy=x+2y$, $3x-4y=1$.

17. $3x-y=5$, $(x+y)^2=17x-y+2$.

18. $20y=9x+13$, $y(13-x)=4(x+2)$.

19. $x^2-3xy=28$, $3x+2y=10$.

20. $x+2y+xy=-2$, $2x-xy=9$.

21. $2x-3y=1$, $x^2+y^2=2x+1$.

22. $x^2+y^2+5x=3$, $5x+4y=2$.

23. $9x^2+6xy+16y^2=4$, $3x-2y=2$.

24. $3x+4y=18$, $x^2+3xy+2y^2=40$.

25. $3x^2+5xy+y^2=-3$, $2x+3y=-1$.

26. $3y-5x=4$, $2x^2+5xy-3y^2=-10$.

27. $x^2+xy=90$, $xy+y^2=10$.

28. $2xy-x^2=24$, $2y^2-xy=30$.

29. $xy+y^2=4$, $x^2-xy+2y^2=8$.

30. $x^2-15=y^2-10=xy$.

31. $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+1}-\frac{1}{y+2}=\frac{1}{12}$.

32. $2xy+y^2=16$, $2x^2-xy=12$.

33. $x^2+2xy=3$, $y^2-xy=4$.

34. $(x+2y)(2x+y)=20$, $y^2+4x(x+y)=16$.

35. $16(x^2-xy)=18(xy-y^2)=1$.

36. $\frac{1}{x}+\frac{1}{y}=(x-4)(y-4)=2$.